FUNCTIONAL ANALYSIS I

Michaelmas Term 2025

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0 Introduction

This is a work in progress for lectures notes to go alongside my lectures on B4.1: *Functional Analysis I*. I am updating and modifying previous versions by Melanie Rupflin, Luc Nguyen, which in turn are based on Hilary Priestley's notes for the course. I am greatful to them all for making these notes available to me. Naturally I am responsible for any errors.

Not having the strict time-limit imposed on a lecture course, the notes tend to go into various (interesting!) digressions and cover additional material which is meant to provide the reader with a "larger and clearer picture". Some parts of the material which are additional and are not covered in the lectures are clearly labeled (as *deep dives*). However, this is not always possible so to know the examinable material you should attend the lectures. I should stress the examinable material is summarised in the syllabus and covered in the lectures – nothing less or more is examinable.

Thanks too to Jan, for the style file for the deep dives, and to Austin for the ducks!

These notes are work in progress and are being constantly improved. I am very grateful to all who have helped me to improve them. Your comments, corrections, but also questions during office hours, are precious.

Please send all your comments and corrections to `stuart.white@maths.ox.ac.uk`. Thank you!

0.1 Overview / Background

Will follow - it's always best to write these last!

0.2 Notation

We will write $\mathbb{F}$ for either $\mathbb{R}$ or $\mathbb{C}$ when it does not matter which of these is the underlying field of our vector space. This course it will rarely matter, but when we focus on operators, particularly from B4.2 onwards it will often be advantageous to work over $\mathbb{C}$, as this is algebraically closed. You'll be familiar with the fact that complex square matrices always have eigenvalues, as the characteristic equation must have a solution over $\mathbb{C}$, but real square matrices need not have any real eigenvalues. This phenomena persists into the infinite dimensional setting: the spectrum of a bounded linear operator on a complex Banach space is always non-empty, and for this reason I prefer to work over the complex field when studying operators. When one is studying Banach spaces in their own right,

Deep Dive

Anything marked as a *Deep Dive* covers material outside of the syllabus. It is only intended for those who are interested and eager to understand things in more depth. It is non-examinable and not necessary for the course. It goes above and beyond the material, often indicating links with other courses and parts of mathematics.

> Even the eager readers should skip those parts on the first reading. More deep dives may appear as I revise the notes. The depth of deep dives may vary considerably from one dive to another.

0.3 Course Synopsis

- Brief recall of material from Part A Metric Spaces and Part A Linear Algebra on real and complex normed vector spaces, their geometry and topology and simple examples of completeness.
- The norm associated with an inner product and its properties.
- Banach spaces, exemplified by ℓ^p, L^p and $C(K)$, and spaces of differentiable functions.
- Finite-dimensional normed spaces, including equivalence of norms and completeness.
- Hilbert spaces as a class of Banach spaces having special properties (illustrations, but no proofs); examples (Euclidean spaces, ℓ^2 and L^2), projection theorem, Riesz Representation Theorem.
- Density. Approximation of functions, Stone-Weierstrass Theorem. Separable spaces; separability of subspaces.
- Bounded linear operators, examples (including integral operators).
- Continuous linear functionals. Dual spaces.
- Statement of the Hahn-Banach Theorem; applications, including density of subspaces and embedding of a normed space into its second dual.
- Adjoint operators.

Contents

1 Normed spaces and Banach spaces

In this section we introduce Banach spaces as complete normed spaces. Our main objective is to give a range of examples. The important special case of Hilbert spaces when the norm is induced from an *inner product* is the subject of Section **??**.

1.1 Definitions and review from metric spaces

The key tool linking used to perform analysis on vector spaces is a *norm*, a suitable notion of distance compatible with the linear algebra structure.

Definition 1.1. Let X be a vector space (over either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$). A *norm* $\|\cdot\| : X \to \mathbb{R}$ is a function with the following properties:

(N1) for all $x \in X$, we have $\|x\| \geq 0$ with $\|x\| = 0 \Leftrightarrow x = 0$;

(N2) for all $x \in X$ and $\lambda \in \mathbb{F}$, we have $\|\lambda x\| = |\lambda|\|x\|$;

(N3) for all $x, y \in X$, the *triangle inequality* $\|x+y\| \leq \|x\| + \|y\|$ holds.

We call a pair $(X, \|\cdot\|)$ a *normed space*. We will often suppress explicit mention of the norm and say 'Let X be a normed space.'

Every norm $\|\cdot\|$ induces a metric

$$d : X \times X \to \mathbb{R}$$

via $d(x,y) := \|x-y\|$ and so all standard notions and properties of a metric space encountered in part A are applicable. Recall:

- Definition of convergence of a sequence (x_n) in X to $x \in X$:

$$x_n \to x \quad \Longleftrightarrow \quad d(x_n, x) \to 0 \quad \Longleftrightarrow \quad \|x_n - x\| \to 0.$$

- A sequence (x_n) in X is called *Cauchy* if:

$$\forall \varepsilon > 0,\ \exists N \text{ such that } \forall n, m \geq N,\ \|x_n - x_m\| < \varepsilon.$$

  This is written as $\|x_n - x_m\| \to 0$ as $m, n \to \infty$. Recall that a metric space X is called *complete* if and only if every Cauchy sequence in X converges to a point in X. For proving completeness it's worth reminding yourself of the following lemma. The proof, left as an exercise, is essentially the same as the last part of the proof of the that Cauchy sequences in $\mathbb{R}$ converge (using the Bolzano–Weierstrass theorem).

Lemma 1.2. *Let (x_n) be a Cauchy sequence in a normed space X. Then the following are equivalent:*

(i) *(x_n) converges,*

(ii) *(x_n) has a convergent subsequence.*

- A function $f : X \to Y$ is *continuous at $x \in X$* if

$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x_0 \in X, \|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \varepsilon.$$

The function f is *continuous* if it is continuous at x for all $x \in X$. These definitions, and in fact all properties of metric spaces,[1] have sequential characterisations: continuous functions are those that preserve limits of sequences. Precisely f is continuous at $x \in X$ if and only if for all sequences (x_n) in X with $x_n \to x$, we have $f(x_n) \to f(x)$. for every $x \in X$ and every sequence (x_n) in X

$$x_n \to x \implies f(x_n) \to f(x).$$

Setting this out in the context of normed spaces, $f : X \to Y$ is continuous if and only if whenever (x_n) is a sequence in X with $\|x_n - x\|_X \to 0$, we have $\|f(x_n) - f(x)\|_Y \to 0$.

- A set $U \subset X$ is open if for every $x_0 \in U$ there exists $r > 0$ such that

$$B_r(x_0) := \{x \in X : \|x - x_0\| < r\} \subset U.$$

Here $B_r(x_0)$ is the open ball centred at x_0 with radius r.

- By definition, a set $F \subset X$ is closed if its complement $X \setminus F$ is open. This can be characterised in terms of sequences: F is closed if and only if whenever (x_n) is a sequence of elements in F which converges to $x \in X$, then $x \in F$, i.e. F contains the limits of all convergent sequences.

The definition of the norm ensures that all the algebraic operations are automatically continuous:

- The scalar multiplication $\mathbb{F} \times X \to X$, $(\lambda, x) \mapsto \lambda x$ is continuous, where $\mathbb{F} \times X$ is given the product metric.
- The addition $X \times X \to X$, $(x, y) \mapsto x + y$ is continuous, where $X \times X$ is given the product metric (below we will give $X \times X$ a norm, but here we only need a metric).
- The norm $\|\cdot\| : X \to \mathbb{R}$ is continuous.

We also recall the appropriate notions of equivalence. First, the equivalence of different norms on the same space:

Definition 1.3. Let X be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are *equivalent* if and only if there exist a constant $C > 0$ so that for all $x \in X$

$$C^{-1}\|x\|_1 \leq \|x\|_2 \leq C\|x\|_1.$$

[1] The topology on a metric space is fully determined by knowledge of which sequences converge to which points. It is for this reason that any property which can be described using the metric can in some way be characterised by sequences. Therefore when I work with normed spaces, or more generally metric spaces, I have a tendancy towards giving sequence based arguments.

Deep Dive

Altthough this course, and Functional Analysis II do not rely on the Part A topology course, I can't resist pointing out that while one can't 'do everything with sequences' in a general topological space, there is a suitable generalisation of a sequence – known as a net – and one can do everything with nets. For example a function $f : X \to Y$ between topological spaces is continuous if and only if whenever $x_i \to x$ is a convergent net in X, then $f(x_i) \to f(x)$ is a convergent net in Y. I've found that the sort of functional analysis arguments I need to do in general topological spaces, work well with nets — sometimes all one needs to do is replace all the n's indexing the sequence with an i indexing a net! Of course you need to know about nets for this; the classic place which I learnt this from is Kelley's 'General topology', and another source is Willard's 'General Topology' (the section on nets is nice and short, but you'll need to dig around elsewher to learn that, for example, a subset is compact if and only if every net has a convergent subnet).

The normed spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are equivalent precisely when the metrics are strongly equivalent in the sense of the metric spaces course.[2] This ensures that all metric notions (open sets, convergent sequences, Cauchy sequences etc) are all the same in both norms.

Secondly, the notion of isomorphism between normed spaces. While there is only one possible notion of an isomorphism between vector spaces –

Definition 1.4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Say X and Y are *isometrically isomorphic* if there exists a surjective linear map $T\colon X \to Y$ with $\|T(x)\|_Y = \|x\|_X$ for all $x \in X$.[3] Say X and Y are *isomorphic* if there exists an isomorphism $T\colon X \to Y$ of vector spaces which is also a homeomorphism of the underlying metric spaces.

We will have much more to say about the linear operators T appearing here in Section **??** and beyond. For now it is useful to know that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ can be (isometrically) isomorphic, without being equivalent (Problem sheet 1 asks you to produce an example).

Deep Dive

You've met a number of theorems in your analysis courses so far that give conditions under which a continuous bijection is a homeomorphism, i.e. has continuous inverse. For example in prelims, a continuous bijection between two intervals has continuous inverse, or a continuous bijection from a compact metric (or topological) space into a metric space (or Hausdorff topological space) has continuous inverse. In functional analysis we have the following consequence of Banach's open mapping theorem, proved in B4.2 using the baire category theorem: if $T\colon X \to Y$ is a bijective continuous linear map between *Banach* spaces, then T^{-1} is continuous, and hence X and Y are isomorphic Banach spaces. This is sometimes called the Banach isomorphism theorem, and decreases the work needed to obtain an isomorphism between Banach spaces.

One of the key objects we study in this course are Banach spaces and linear maps between such spaces.

Definition 1.5. A normed space $(X, \|\cdot\|)$ is a Banach space if it is complete, i.e. if every Cauchy sequence in X converges.

A normed space is complete if and only if absolute convergence of series implies convergence of series:

Proposition 1.6. *Let X be a normed space. Then the following are equivalent*

(i) X is a Banach space,

(ii) Absolute convergence of series implies convergence, i.e. for sequences (x_n) in X and the corresponding partial sums $s_n := \sum_{k=1}^n x_k$ we have

$$\sum_{i=1}^{\infty} \|x_n\| < \infty \quad \Rightarrow \quad s_n \text{ converges to some } s \in X.$$

Proof. (i) $\Longrightarrow$ (ii): Suppose X is complete and $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Then, for $s_n = \sum_{k=1}^n x_n$, the sequence (s_n) is Cauchy as for $m > n \geq N$

$$\|s_n - s_m\| = \|\sum_{k=n+1}^{m} x_k\| \leq \sum_{k=n+1}^{m} \|x_k\| \leq \sum_{k=N+1}^{\infty} \|x_k\| \to 0 \text{ as } N \to \infty.$$

[2] You might be suspicious of the terminolgy: normed spaces are called equivalent when their underlying metric structures are strongly equivalent. But while in the generality of metric spaces, equivalence of two metrics d_1 and d_2 on X – defined in Part A metric spaces to mean that the identity map Id_X is a homeomorphism – is not the same notion as strong equivalence, for normed spaces it is. This will all follow from equivalence between continuity and boundedness of linear maps in Section **??**.

[3] Such a map T is necessarily injective, so T is an isomorphism of vector spaces which is also an isometry.

As X is complete we thus obtain that s_n converges to some element $s \in X$.

(ii) $\implies$ (i): Suppose (ii) holds and let (x_n) be a Cauchy sequence. Select a subsequence x_{n_j} so that

$$\|x_{n_j} - x_{n_{j+1}}\| \leq 2^{-j},$$

where the existence of such a subsequence is ensured by the fact that (x_n) is Cauchy. Then $\sum_{j=1}^{\infty} \|x_{n_{j+1}} - x_{n_j}\| \leq 1 < \infty$ so (ii) ensures that $\sum_{j=1}^{\infty}(x_{n_{j+1}} - x_{n_j})$ converges. Hence $x_{n_k} = x_{n_1} + \sum_{j=1}^{k-1}(x_{n_{j+1}} - x_{n_j})$ converges, so (x_n) has a convergent subsequence and must thus, by Lemma 1.2, itself converge. $\square$

1.2 Examples

$(\mathbb{R}^n, \|\cdot\|_p)$ and $(\mathbb{C}^n, \|\cdot\|_p)$, $1 \leq p \leq \infty$. Consider $\mathbb{R}^n$, or $\mathbb{C}^n$, equipped with

$$\|x\|_p := \Big(\sum_i |x_i|^p\Big)^{1/p} \text{ for } 1 \leq p < \infty$$

respectively

$$\|x\|_\infty := \sup_{i \in \{1,\dots,n\}} |x_i|.$$

One can show that these are all norms, with the challenging bit being the proof of the triangle inequality

$$\|x+y\|_p = \Big(\sum_i |x_i + y_i|^p\Big)^{1/p} \leq \|x\|_p + \|y\|_p,$$

which is is the finite dimensional version of Minkowski's inequality.

Warning. This inequality does not hold if we were to extend the definition of $\|\cdot\|_p$ to $0 < p < 1$, and hence the above expression does not give a norm on $\mathbb{R}^n$ if $p < 1$.

We will write ℓ_n^p for the normed space $(\mathbb{R}^n, \|\cdot\|_p)$. A useful property to deal with the p norms $1 \leq p \leq \infty$ (and their generalisations to sequence and functions spaces) is Hölder's inequality (which we proved in much more generality in the Integration course – see Proposition 1.12).

Lemma 1.7 (Hölder's inequality in $\mathbb{R}^n$)**.** *For $1 \leq p, q \leq \infty$ with*

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{1}$$

we have that for any $x, y \in \mathbb{C}^n$

$$\Big|\sum_{i=1}^n x_i y_i\Big| \leq \|x\|_p \|y\|_q.$$

In (1) we use the convention that $\frac{1}{p} = 0$ for $p = \infty$, and one often calls numbers $p, q \in [1, \infty]$ satisfying (1) *Hölder conjugate exponents*.

Remark. As you will show on Problem sheet 1, we have that for all $1 \leq p < \infty$

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p} \|x\|_\infty.$$

Hence the ∞-norm is equivalent to every p-norm and thus, by transitivity, we have that $\|\cdot\|_p$ is equivalent to $\|\cdot\|_q$ for every $1 \leq p, q \leq \infty$. In fact as we will show in Section **??**, all norms on finite dimensional spaces are equivalent.

Deep Dive

With some small exceptions (which I neglected to mention in lectures), the spaces ℓ^p_n are not isometrically isomorphic as p varies. The first exception is if $n = 1$: then all the norms are the same! The only other exception is if $n = 2$, when ℓ^1_2 is isometrically isomorphic to ℓ^∞_2, via the map $(x,y) \to (x+y, x-y)$. In a deep dive below we discuss to how see this sort of thing for the sequence spaces ℓ^p, and come back to ℓ^1_n and ℓ^∞_n for $n \geq 3$.

Sequence spaces ℓ^p and c_0. An infinite dimensional analogue of $(\mathbb{R}^n, \|\cdot\|_p)$, respectively $(\mathbb{C}^n, \|\cdot\|_p)$ are the spaces of sequences ℓ^p, $1 \leq p \leq \infty$, where for $1 \leq p < \infty$

$$\ell^p := \left\{ (x_j)_{j\in\mathbb{N}} : \sum_{j=1}^{\infty} |x_j|^p < \infty \right\}$$

equipped with $\|\cdot\|_p$ where for $1 \leq p < \infty$

$$\|x\|_{\ell^p} = \left(\sum_{j=1}^{\infty} |x_j|^p \right)^{1/p},$$

while ℓ^∞ denotes the space of bounded sequences, equipped with

$$\|(x_j)\|_\infty := \sup_j |x_j|.$$

For any $1 \leq j \leq \infty$ we have that ℓ^p is a normed space (where we define addition and scalar-multiplication component-wise). Again the main difficulty is to obtain Minkowski's inequality, which is precisely the triangle inequality.

Example 1.8. ℓ^p is complete for $1 \leq p \leq \infty$.

This follows the standard procedure for showing completeness. Given a Cauchy sequence (x_n) in a normed space X:

1. identify a candidate x for $\lim x_n$;
2. show that $x \in X$;
3. show $\|x - x_n\|_X \to 0$ as $n \to \infty$.

Often, as in the proof below, which we recall from metric spaces, steps 2 and 3 can be performed simultaneously. The $p = \infty$ case is easier (and a special case of the Banach space $\mathscr{F}_b(\Omega)$ below, by taking $\Omega = \mathbb{N}$.)

Proof for $1 \leq p < \infty$. Let $(x^{(n)})$, be a Cauchy-sequence in ℓ^p and write $x^{(n)} = (x_j^{(n)})_{j=1^\infty}$. As for every $j \in \mathbb{N}$

$$|x_j^{(n)} - x_j^{(m)}| \leq \|x^{(n)} - x^{(m)}\|_2 \to 0$$

as $m, n \to \infty$, the sequence $(x_j^{(n)})$ is Cauchy in $\mathbb{F}$ so converges, say $x_j^{(n)} \to x_j$.

Fix $\varepsilon > 0$. Then there exists N so that for all $n, m \geq N$

$$\|x^{(n)} - x^{(m)}\|_p \leq \varepsilon.$$

Thus for every $K \in \mathbb{N}$ and for all $n \geq N$ we have that

$$\sum_{j=1}^{K} |x_j^{(n)} - x_j|^p = \lim_{m\to\infty} \sum_{j=1}^{K} |x_j^{(n)} - x_j^{(m)}|^p \leq \varepsilon^p.$$

Letting $K \to \infty$, it follows that for all $n \geq N$, we have $x^{(n)} - x \in \ell^p$, and so $x \in \ell^p$ (step 2), and $\|x^{(n)} - x\|_p \leq \varepsilon$ for $n \geq N$, so that $x^{(n)} \to x$ in ℓ^p. □

Again Hölder's inequality is valid. That is for every $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and any $(x_j) \in \ell^p$ and $(y_j) \in \ell^q$ we have that $\sum x_j y_j$ converges, i.e. the pointwise product $xy \in \ell^1$ and

$$|\sum_j x_j y_j| \leq \|(x_j)\|_p \|(y_j)\|_q.$$

Again this is a special case of Proposition 1.12 from Integration, taking the measure space to be $\mathbb{N}$ equipped with counting measure.

We will sometimes also consider the subspace

$$c_0 := \{(x_n) \in \ell^\infty : x_n \to 0\}$$

of ℓ^∞, which is closed[4] and hence, when equipped with the ℓ^∞- norm a Banach space (see Proposition 1.16 below)

Note that all of these sequence spaces contain a common subspace c_{00} – the collection of sequences which are eventually zero. You should check that c_{00} is dense in c_0, and in ℓ^p for $1 \leq p < \infty$. Accordingly it can not be a Banach space in any of the p-norms (by Proposition 1.16 below).

Deep Dive

In fact there is no norm on c_{00} under which it is a Banach space.

Theorem. *No infinite dimensional Banach space X can have a countable* Hammel basis.

A *Hammel basis* is what up to now we've just called a basis, i.e. a linearly independent spanning set. This is a purely algebraic notion so $\mathscr{S}$ is a Hammel basis for X when no non-trivial *finite* linear combination of elements of $\mathscr{S}$ can be zero, and every element of X can be written as a finite linear combination of elements of $\mathscr{S}$. While in a normed space we are allowed to consider infinite sums, these are not used to define Hammel bases. Certainly the polynomials have a countably infinite Hamel basis and so can not be a Banach space under any norm.

For background, by a Zorn's lemma argument every vector space has a Hammel basis. Zorn's lemma is a tool equivalent to the axiom of choice, which will appear in some other deep dives, but is definitely non-examinable. Zorn's lemma will be described in B1.2 (Set Theory) which will show that every vector space having a Hammel basis is another reformulation of the axiom of choice (that course will use the language of basis, rather than Hammel basis, as the vector spaces there do not come equipped with norms).

The normal, and in my view best way, to prove that no Banach space can have a countably infinite Hammel basis is through the Baire category theorem, which will be proved in B4.2 Functional Analysis 2. This states that a complete metric space is never a countable union of nowhere dense subsets.[a] We will see in Section **??** that finite dimensional subspaces of normed spaces are always closed, and it is an easy exercise to check that a proper closed subspace of a normed space is nowhere dense. Then the result follows. On problem sheet 2 we will see an alternative, slightly messier proof in Section C, using Riesz's lemma from Section **??**.

[4] This is very similar to Problem sheet 1, B.1(a).

This result shows that Hammel bases, while they generally exist, are unlikely to be all that useful for working with Banach spaces. Even for spaces such as ℓ^1 or c_0, where we clearly only need countably many bits of data to specify elements, do not have countable bases. Instead, one needs the appropriate analytic notion of a basis – *Schauder bases* – to develop a satisfactory theory for Banach spaces. These appear in more detail at the end of the C4.1 course, but they are perfectly accessible to you now. A good book taking a basis first approach, and so developing interesting properties of the sequence spaces is Carothers 'A short course in Banach space theory'. Definitely the first 6 chapters of this book are very readable along side this course - but from chapter 3 onwards go in different direction.

[a] A nowhere dense set is a set whose closure has empty interior.

When I'm trying to build counter examples, my first thought is to check the finite dimensional case, and assuming that doesn't work then I tend to look for an example using sequences.

Deep Dive

While for $n \in \mathbb{N}$ fixed, all the n-dimensional normed spaces ℓ^p_n are equivalent, this fails in infinite dimensions. Also, just as in the finite dimensional case, by looking at the geometry of the unit balls one can see that no pair of these spaces is isometrically isomorphic. One thing you can look at here is the *modulus of convexity*, which quantifies the convexity of the unit ball. For a Banach space X, define the modulus of convexity $\delta_X : [0,2] \to [0,1]$ by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x,y \in X,\ \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\right\}.$$

This measures how far the average of two points x and y of distance at least ε on the sphere must get pushed inside the unit ball. (There's some nice pictures to be drawn here; if someone draws them I'll be happy to include). You should be able to see that for ℓ^∞ and ℓ^1 we have $\delta = 0$. The modulus of convexity of ℓ^2, and indeed any Hilbert space, is $\delta(\varepsilon) = 1 - \sqrt{1-\varepsilon^2/4} = \varepsilon^2/8 + O(\varepsilon^4)$ as $\varepsilon \to 0$.footnoteIt's small ε we care about, otherwise we wouldn't have used the notation ε. A two-dimensional geometric argument shows that this is the upper bound of the modulus of convexity for any normed space X; in this sense Hilbert space's have the "convexest" unit ball. For other p, we have

$$\delta_{\ell^p}(\varepsilon) = \begin{cases} (p-1)\frac{\varepsilon^2}{8} + o(\varepsilon^2), & 1 < p \leq 2; \\ \frac{\varepsilon^p}{p2^p}, & 2 \leq p < \infty. \end{cases}$$

While the modulus of convexity doesn't distinguish ℓ^∞_n and ℓ^∞_n, for $n \geq 3$, you can see that nevertheless they are not isometrically isomorphic from the geometry of the unit balls: the ball of ℓ^1_n has $2n$ extreme points, and the ball of ℓ^∞_n has 2^n extreme points.[a]

Showing that in infinite dimensions none of the sequence spaces ℓ^p or c_0 are isomorphic is more challenging. By section **??** we will know that ℓ^∞ is not isomorphic to any of the other spaces — it is too big (not separable to be precise). By the end of the course, we will know that c_0, ℓ^1 are not isomorphic and also not isomorphic to any other ℓ^p space. In fact all these spaces are pairwise non-isomorphic. This is much harder, and we won't prove it in any of the functional analysis courses here. (The way this is normally done is through Pitt's theorem, that says that for $1 \leq p < q < \infty$, any bounded linear map (see Section **??**) $T : \ell^q \to \ell^p$ is compact (a concept that will be defined in B4.2). Taking this fact for granted (a proof can be found in Chapter 2 of Albaic and Kalton's "Topics in Banach Space Theory") you should be able to deduce that all

these sequence spaces are pairwise non-isomorphic by the end of B4.2.

[a] I'll let you formalise or look up the definition of an extreme point. A picture ought to work in finite dimensions, but in infinite dimensions some unit balls need not have extreme points, for example c_0. I will make further deep dive remarks on this later in the notes.

Function spaces with supremum-norm

The supremum norm is often used to make spaces of bounded functions complete.

Example 1.9. Let Ω be any set. Then

$$\mathscr{F}^b(\Omega) := \{f : \Omega \to \mathbb{F} \text{ bounded}\}$$

is a Banach space under the norm $\|f\|_{\sup} = \sup_{x\in\Omega} |f(x)|$. (We will often drop the subscript sup on these norms).

Proof of completeness of $\mathscr{F}^b(\Omega)$. This is another example of the three step process. If (f_n) is Cauchy in $\mathscr{F}^b(\Omega)$ then for each $x \in \Omega$, $(f_n(x))$ is Cauchy so converges to $f(x) \in \mathbb{F}$, say. Fixing $\varepsilon > 0$, there is $N \in \mathbb{N}$ with $|f^{(n)}(x) - f^{(m)}(x)| \leq \varepsilon$ for $m,n \geq N$ and all $x \in \Omega$. Taking limits as $m \to \infty$, $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in \Omega$ and $n \geq N$. Therefore f is bounded, and $\|f_n - f\| \leq \varepsilon$. Accordingly $\mathscr{F}^b(\Omega)$ is complete. □

For a metric[5] space K, the vector space $C_b(K) := \{f : K \to \mathbb{F} \text{ continuous and bounded}\}$ is closed in $\mathscr{F}^b(K)$,[6] and hence $C_b(K)$ is a Banach space with the sup norm (see Proposition 1.16). If K is compact, then all continuous functions are bounded, and we write $C(\Omega) := \{f : \Omega \to \mathbb{F} \text{ continuous}\} = C_b(\Omega)$.

Example 1.10. Let K be a compact metric space. Then $C(K)$ is a Banach space with the sup-norm.

Similarly, on spaces of differentiable functions (with bounded derivatives) such as $C^1([0,1])$ – the space of functions on $[0,1]$ which have continuous derivatives (including at the end points), to get completeness we will need norms that are built using the sup norm of both the function and its derivative. Indeed, since $C^1([0,1])$ is dense in $C[0,1]$ (in fact the smaller set of polynomials is dense in $C[0,1]$ – see Section **??**), it is not closed in the sup-norm and so is not a Banach space with respect to $\|\cdot\|_{\sup}$. Instead we use a standard trick for creating a norm on spaces like this; take the sum of two norms that we want to be able to control. Then a sequence which is Cauchy in the sum of the norms, will necessarily be Cauchy in each norm.

Example 1.11. $C^1([0,1])$ is a Banach space with $\|f\|_{C^1} := \|f\|_{\sup} + \|f'\|_{\sup}$

Proof. Suppose (f_n) is a Cauchy sequence in $C^(([0,1])$. Then (f_n) is $\|\cdot\|_{\sup}$-Cauchy so converges to some $f \in C([0,1])$ and (f_n') is also $\|\cdot\|_{\sup}$-Cauchy so converges to some $g \in C([0,1])$. Now prelims comes to the rescue: since (f_n) converges uniformly to f and each f_n' is differentiable and f_n' converges uniformly to g, it follows that f is differentiable and $f' = g$. Hence $f \in C^(([0,1])$, and then $\|f_n - f\|_{C^1} \to 0$. □

Deep Dive

The map $f \mapsto (f', f(0))$ gives an isomorphism[a] from $C^1([0,1])$ to $C([0,1]) \times \mathbb{R}$ (we will discuss norms on the Cartesian product below. In fact, using a Schauder basis for $C([0,1])$ it is possible to show that $C([0,1]) \cong C([0,1]) \times \mathbb{R}$ though the isomorphism can not be isometric,[b] so that as Banach spaces $C^1([0,1])$ and $C([0,1])$ are isomorphic. But in applications we would most likely care about how our elements are realised as functions: so while these Banach spaces are abstractly isomorphic, it makes sense to understand

[5] this works fine for a topological space

[6] this is a consequence of the fact that a uniform limit of continuous functions is continuous — a 3ε or $\varepsilon/3$ argument depending on your taste.

them separately.

[a]This map is bounded and bijective between Banach spaces, so an isomorphism from Banach's isomorphism theorem; though you could show directly that the inverse map is bounded.

[b]We're implicitly working with reals here, so you can use the geometry of the unit ball: $C([0,1])$ has two extreme points, the constant functions ± 1, while the unit ball of $C([0,1]\times\mathbb{R}$ with the ℓ^2-norm has more.

Function spaces L^p, $1 \le p \le \infty$

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space (considering examples like Ω an interval in $\mathbb{R}$ or a measurable subset of $\mathbb{R}^n$ with Lebesgue measure will be more than sufficient for the course). Consider for $1 \le p < \infty$ the space of functions

$$\mathscr{L}^p(\Omega) := \Big\{ f : \Omega \to \mathbb{R} \text{ (or } \mathbb{C}) \text{ measurable so that } \int_\Omega |f|^p dx < \infty \Big\}$$

respectively

$$\mathscr{L}^\infty := \Big\{ f : \Omega \to \mathbb{R} \text{ (or } \mathbb{C}) \text{ measurable so that } \exists M \text{ with } |f| \le M \text{ a.e.} \Big\}.$$

This is not a course on Lebesgue integration, so typically in examples we will only work with measurable functions; you may assume that all functions you encounter are measurable when needed (though don't let that stop you briefly reminding yourself why). But of course not all measurable functions are integrable and that indeed for a general measurable function the integral might not even be defined, so justification is needed to consider integrals in general. However we also recall that the integral of a non-negative functions f is always defined though might be infinite.

We equip these spaces with

$$\|f\|_p := \Big(\int_\Omega |f|^p dx \Big)^{1/p} \text{ for } 1 \le p < \infty$$

respectively

$$\|f\|_\infty := \text{ess sup}|f| := \inf\{M > 0 : |f| \le M \text{ a.e. }\}.$$

We note that $\|\cdot\|_p$ is only a seminorm on $\mathscr{L}^p$ with $\|f-g\|_p = 0$ if and only if $f = g$ a.e. We can hence turn $(\mathscr{L}^p, \|\cdot\|)$ into a normed space by taking the quotient with respect to the equivalence relation

$$f \sim g \iff f = g \text{ a.e..}$$

The resulting quotient space

$$L^p(\Omega) := \mathscr{L}^p / \sim \text{ equipped with } \|\cdot\|_p$$

is one of the most important spaces of functions in the modern theory of PDE, and will be further developed in the course C4.3 Functional analytic methods for PDEs. Recall the following two key inequalities for L^p spaces; the first giving the triangle inequality, and the second will be crucial later in the course when we examine the dual spaces of $L^p(\Omega)$.

Proposition 1.12 (Minkowski and Hölder). *The Minkowski-inequality, which is the triangle inequality for L^p holds: for $f, g \in L^p(\Omega)$, we have*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

- *Hölder's inequality holds: If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$ then their product fg is integrable with*

$$\Big| \int_\Omega fg dx \Big| \le \|f\|_p \|g\|_q.$$

None of the L^p norms are equivalent, though when Ω has positive and finite measure, we can estimate the L^p norm of functions by their L^q norm if $p < q$ and we have

$$L^\infty(\Omega) \subsetneqq L^q(\Omega) \subsetneqq L^p(\Omega) \subsetneqq L^1(\Omega) \text{ for any } 1 < p < q < \infty. \tag{2}$$

As an example consider $\Omega = (0,2) \subset \mathbb{R}$ and $p = 2$, $q = 4$. Adding in a multiplication by the constant function $g = 1$ we can estimate, using Hölder's inequality,

$$\|f\|_{L^2}^2 = \int |f|^2 \cdot 1 dx \leq \||f|^2\|_{L^2} \|1\|_{L^2} = \Big(\int_0^2 f^4\, dx\Big)^{1/2} \cdot \Big(\int_0^2 1\, dx\Big)^{1/2} = \sqrt{2}\|f\|_{L^4}^2,$$

so we get $\|f\|_{L^2} \leq \sqrt{2}\|f\|_{L^4}$ and in particular that every $f \in L^4([0,2])$ is also an element of $L^2([0,2])$. The general case is discussed on the first problem sheet.

Warning.
- Note that the inclusions of the function spaces $L^p(\Omega)$ for sets Ω with bounded measure are the "other way around" compared with the inclusions of the sequence spaces ℓ^p.
- The inclusion (2) is wrong for unbounded domains, e.g. the constant function $f = 1$ is an element of $L^\infty(\mathbb{R})$ but isn't contained in any $L^p(\mathbb{R})$, $1 \leq p < \infty$.

Remark. In practice it is can be useful to extend $\|\cdot\|_{L^p}$ to a function from the space of all (measurable) functions to $[0,\infty) \cup \{\infty\}$ by simply setting $\|f\|_{L^p} = \infty$ if $\int |f|^p = \infty$ (respectively for $p = \infty$ if $f \notin L^\infty$), and we note that also with this 'abuse of notation' the triangle and Hölder-inequality still hold (with the convention that $0 \cdot \infty = 0$ for Hölder's inequality). Similarly we can extend $\|\cdot\|_p$ to a function that maps all sequences to $[0,\infty) \cup \{\infty\}$ but we stress that while this notation/convention can be useful and used in the literature, these functions into $[0,\infty) \cup \{\infty\}$ are not norms as a norm is by definition a function into $[0,\infty)$.

Example 1.13. $(L^\infty(\Omega), \|\cdot\|_{L^\infty})$ is a Banach space.

The proof is more or less similar to the proof of completeness for $\mathscr{F}_b(\Omega)$, or a direct proof of completeness for $C(K)$ in the supremum norm, except that we have to take care of the almost everywhere nature of things.

Proof. Let (f_n) be a Cauchy sequence in $L^\infty(\Omega, \mathbb{R})$. For each $k \in \mathbb{N}$, there exists N_k such that

$$\|f_n - f_m\|_{L^\infty} \leq \frac{1}{k} \text{ for all } n, m \geq N_k.$$

This means that, for each k and $m, n \geq N_k$ there is a null subset $Z_{k,m,n}$ of Ω such that

$$|f_n(x) - f_m(x)| \leq \frac{1}{k} \text{ for } x \in \Omega \setminus Z_{k,m,n}.$$

Let $Z = \cup_k \cup_{n,m \geq N_k} Z_{k,n,m}$, which, as a countable union of null set, is null. Then,

$$|f_n(x) - f_m(x)| \leq \frac{1}{k} \text{ for all } n, m \geq N, x \in \Omega \setminus Z. \tag{3}$$

So for almost all $x \in \Omega$, $(f_n(x))$ is Cauchy, and hence converges to some $f(x)$.

Being an almost everywhere limit of measurable functions, f is measurable. Sending $m \to \infty$ while keeping n fixed in (3) we get

$$|f_n(x) - f(x)| \leq \frac{1}{k} \text{ for all } n \geq N_k, x \in \Omega \setminus Z.$$

This shows that $\|f_n - f\|_{L^\infty} \leq \frac{1}{k}$ for all $n \geq N_k$. This implies on one hand that $f_n - f$ and hence f belong to $L^\infty(\Omega)$ and on the other hand that $f_n \to f$ in $L^\infty(\Omega)$. $\square$

Example 1.14. L^p is complete for $1 \leq p < \infty$.

We give two proofs. Firstly we consider the proof lectured in part A integration,[7] and show how this really fits into the abstract framework of proving completeness by showing that absolutely convergence series converge.

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence in $L^p(\Omega)$ with $\sum_n \|f_n\|_p < \infty$. Define $g_n = \sum_{r=1}^{n} |f_r|$. This gives an increasing sequence of non-negative measurable functions which converges to $g = \sum_{n=1}^{\infty} |f_n|$ (g can of course take the value ∞ whenever this sum diverges). By Minkowski

$$\int g_n^p = \|g_n\|_p^p \leq \left(\sum_{r=1}^{n} \|f_n\|_p \right)^p \leq \left(\sum_{n=1}^{\infty} \|f_n\|_p \right)^p < \infty.$$

Therefore by the monotone convergence theorem g^p is integrable, i.e. $g \in L^p$, and hence g is finite almost everywhere. That is the sum $\sum_{n=1}^{\infty} f_n$ is absolutely convergent almost everywhere, and so converges almost everywhere, say to f.[8] Moreover, applying the triangle inequality pointwise $|f|^p \leq g^p$, so $f \in L^p$ by comparison.[9]

Finally, another application of the triangle inequality gives

$$|f - \sum_{r=1}^{n} f_r|^p \leq \left(\sum_{r=n+1}^{\infty} |f_r| \right)^p \leq g^p,$$

so the dominated convergence theorem gives $\|f - \sum_{r=1}^{n} f_r\|_p^p \to 0$, and $\sum_{n=1}^{\infty} f_n = f$ in L^p.[10] □

The second proof finds the limit through finding by means of finding a subsequence which is convergent almost everywhere. Since we didn't lecture convergence in measure in the integration course, I'm reworking this proof to give some background as the ideas may be useful to you elsewhere.

Deep Dive

We should also discuss potential isomorphisms between all the L^p-spaces and all the ℓ^p spaces. Firstly none of the L^p spaces are isometrically isomorphic; the same modulus of convexity formula in one of the deep dives above works for L^p.

$L^2(\Omega, \mathscr{F}, \mu)$ is a Hilbert space, and as we will learn these are determined up to isometric isomorphism by the size of an orthonormal basis. In most examples of interest to us, $(\Omega, \mathscr{F}, \mu)$ has just the right 'size' to be separable (have a countable dense subset; see Section **??**), and in this case it will be isometrically isomorphic to ℓ^2. This will always be the case when $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable subset (with non zero measure) equipped with Lebesgue measure.[a] As an example you might well be able to guess an isometric isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$.

There is more to come here!

[a] In general you need that Ω is σ-finite and $\mathscr{F}$ is the completion of a countably generated σ-algebra.

Some incomplete spaces

[7] The case of L^p was lectured, but only the case of L^1 is in the lecture notes.

[8] This is Step 1 of the process for completeness (in this absolute convergence framework) by providing a candidate limit.

[9] Step 2: the limit is in the space it should be in.

[10] The expression in L^p relating to this sum means that we have justified convergence of the sum in the norm on L^p, as required for Step 3.

Example 1.15. We can construct many examples of non-complete spaces by equipping a well known space such as C_b, C^1, ℓ^p, L^p with the 'wrong' norm, or by choosing a subspace of a Banach space that is not closed. As an example we show that $C([0,1])$ equipped with $\|f\|_{L^1} = \int_0^1 |f|dx$ is not complete (which is actually an exercise from the metric spaces course and so I won't lecture it).

Proof. We give three proofs: one by direct argument, one via Corollary 1.6, and finally through density.

1. Let
$$g_n(x) = \begin{cases} (2x)^n & \text{for } x \in [0,1/2), \\ 1 & \text{for } x \in [1/2,1]. \end{cases}$$
For $n < m$, we have
$$\|g_n - g_m\|_{L^1} = \int_0^{1/2} [(2x)^n - (2x)^m]\, dx = \frac{1}{2(n+1)} - \frac{1}{2(m+1)},$$
so (g_n) is Cauchy. On the other hand, (g_n) is a decreasing sequence of non-negative functions which is bounded from above by 1. Its pointwise limit is the characteristic function of the interval $[1/2,1]$. By Lebesgue's dominated convergence theorem, g_n converges to $\chi_{[1/2,1]}$ in L^1 and there is no continuous function which is equal to $\chi_{[1/2,1]}$ almost everywhere,[11], and hence not in $C([0,1])$. In other words (g_n) has no limit in $C([0,1])$.[12]

2. For
$$f_n(x) := \begin{cases} 1 - n^2 x & \text{for } x \in [0, \frac{1}{n^2}] \\ 0 & \text{else} \end{cases}$$
we have that $\|f_n\|_1 = \frac{1}{2n^2}$ so $\sum \|f_n\|_{L^1}$ converges. However $\sum f_n$ cannot converge to an element of $C([0,1])$. Indeed suppose, seeking a contradiction, that $\sum f_n \to f$ converges in L^1 to a function $f \in C([0,1])$. Then, as continuous functions on compact sets are bounded, there exists some $M \in \mathbb{R}$ so that $f \leq M$ on $[0,1]$. Hence choosing $N \in \mathbb{N}$ so that $N \geq 2(M+1)$ we obtain that for any $n \geq N$ and any $x \in [0, \frac{1}{2N^2}]$
$$\sum_{j=1}^{n} f_j(x) - f(x) \geq \sum_{j=1}^{N} \frac{1}{2} - f(x) \geq N/2 - M \geq 1$$
and thus in particular $\|\sum_{j=1}^{n} f_j - f\|_1 \geq \frac{1}{2N^2} \not\to 0$.

3. We know from part A integration that $C([0,1])$ is a proper dense subspace of $L^1([0,1])$, so can not be complete (by Proposition 1.16).

□

1.3 Constructions

We end this section with a brief collection of ways to construct new normed spaces from existing examples, and when this preserves completeness.

[11] $\chi_{[1/2,1]}$ is almost everywhere continuous, but not almost everywhere equal to a continuous function

[12] Here we are using the fact that we know $C([0,1])$ is a subspace of $L^1([0,1])$ so limits are unique. When you did this exercise in the metric spaces course, this answer would not have been sufficient as we didn't have the space $L^1([0,1])$ to work with. In part A metric spaces you where supposed to deal with this by showing barehands that there is no continuous function which can arise as the L^1 limit.

Subspaces We first note that for any given subspace Y of a normed space $(X, \|\cdot\|)$ we obtain a norm on Y simply by restricting the given norm to Y. For the resulting normed space $(Y, \|\cdot\|)$ we have

Proposition 1.16. *Let X be a Banach space, $Y \subset X$ a subspace. Then*

$$(Y, \|\cdot\|) \text{ is complete } \Leftrightarrow Y \subset X \text{ is closed}.$$

Proof. Suppose Y is complete, and (y_n) is a sequence in Y with $y_n \to x \in X$. As (y_n) converges in X, it is Cauchy. Therefore by completeness it converges to some $y \in Y$. Hence $x = y \in Y$ by uniqueness of limits and Y is closed.

Conversely, suppose Y is closed in X and let (y_n) be a Cauchy sequence in Y. By completeness of X, it follows that there exists $x \in X$ with $y_n \to x \in X$. But as Y is closed we must have that $x \in Y$ and hence that (y_n) converges in Y. Therefore Y is complete. □

Direct sums Given two normed spaces X and Y we can define a norm on $X \times Y$ e.g. by

$$\|(x,y)\|_2 = (\|x\|^2 + \|y\|^2)^{1/2} \tag{4}$$

or more generally using any of the p-norms on $\mathbb{R}^2$ to define

$$\|(x,y)\|_p := \|(\|x\|, \|y\|)\|_p = (\|x\|^p + \|y\|^p)^{1/p} \text{ respectively } \|(x,y)\|_\infty := \max(\|x\|, \|y\|)$$

where here and in the following we simply write $\|\cdot\|$ instead of $\|\cdot\|_X$ and $\|\cdot\|_Y$ if it is clear from the context what norm we are using. As all (the ℓ^p)-norms on $\mathbb{R}^2$ are equivalent, it follows that all the norms $\|(x,y)\|_p$ are equivalent on $X \times Y$. We tend to write $X \oplus_p Y$ for these spaces.

We note that for all of these norms on $X \times Y$ we obtain that $X \times Y$ is again a Banach space if both X and Y are Banach spaces. If X and Y are inner product spaces then one uses in general the norm (4) as for this choice of norm also the product $X \times Y$ will again be a inner product space with inner product $((x,y),(x',y')) = (x,x')_X + (y,y')_Y$, while none of the norms with $p \neq 2$ preserve the structure of an inner product space.

Deep Dive

We can also consider countable direct sums. Given normed spaces $(X_n)_{n=1}^\infty$, and $1 \leq p \leq \infty$, one can form the ℓ^p-direct sums. For $p = \infty$, let

$$X_\infty = \{(x_n) : x_n \in X_n,\ \sup \|x_n\| < \infty\}$$

with the norm $\|(x_n)\|_\infty = \sup \|x_n\|$. For $1 \leq p < \infty$ let

$$X_p = \{(x_n) : x_n \in X_n : \sum_{n=1}^\infty \|x_n\|^p < \infty\}$$

with the norm $\|(x_n)\|_p = (\sum_{n=1}^\infty \|x_n\|^p)^{1/p}$.

These spaces are often written as $(\bigoplus_{n=1}^\infty X_n)_p$.

It's a good exercise in seeing if you understand the proofs that ℓ^p forms a Banach space to check these are norms, and that if each X_n is complete so too our the spaces X_p. This time of course all these norms will in general give rise to pairwise non-isomorphic spaces (as can be seen by taking each $X_n = \mathbb{F}$ when you get back the classical sequence spaces ℓ^p). Now you can start asking what sort of spaces you get if you take an infinite ℓ^p product say of a sequence of L^{q_n} spaces!

Sums of subspaces If $X_1, X_2 \subset X$ are subspaces of a normed space X then also

$$X_1 + X_2 := \{x_1 + x_2 : x_1 \in X_1, x_2 \in X_2\}$$

is again a subspace of X, but beware. Just because X_1 and X_2 are closed in X, it does not necessarily follow that $X_1 + X_2$ is closed; see example sheet 1 (question C1) for an example.

Quotients We saw the process of taking quotients to pass from the semi-norm on $\mathscr{L}^p$ to the normed space L^p above. This works generally. Given a vector space X and a semi-norm $|\cdot|$ on X, i.e. a function $|\cdot| : X \to [0,\infty)$ satisfying (N2) and (N3), we can consider the quotient space X/X_0 where $X_0 := \{x \in X : |x| = 0\}$. Then one can define a norm on X/X_0 by defining $\|x + X_0\| := |x|$, see problem sheet 1 for details.

Deep Dive

There are many reasons to be interested in quotient spaces more generally. Suppose X is a normed space, and Y is a subspace of X, when can we put a norm on X/Y? The solution is to define

$$\|x + Y\| = \inf\{\|x + y\| : y \in Y\} = \inf\{\|x - y\| : y \in Y\} = d(x, Y),$$

In general this is only a seminorm as if $\|x + Y\| = 0$, then there is a sequence $y_n \in Y$ with $\|x + y_n\| \to 0$. Noting that $-y_n \in Y$, it follows that $\|x + Y\| = 0$ if and only if x is in the closure $\overline{Y}$ of Y. In this way we get a norm on X/Y precisely when Y is closed.[a].

This will be explored further in C4.1, but it is nice to know that the quotient of a Banach space by a closed subspace is again a Banach space (this is normally proved by showing absolute convergence implies convergence).

[a] When Y is not closed, you could follow the construction of taking a further quotient of X/Y by the null space of the seminorm. You can check that this gives the same thing as considering the quotient $X/\overline{Y}$, so in practise we only consider quotients by closed subspaces.