FUNCTIONAL ANALYSIS I

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0 Introduction

This is a work in progress for lectures notes to go alongside my lectures on B4.1: *Functional Analysis I*. I am updating and modifying previous versions by Melanie Rupflin, Luc Nguyen, which in turn are based on Hilary Priestley's notes for the course. I am greatful to them all for making these notes available to me. Naturally I am responsible for any errors.

Not having the strict time-limit imposed on a lecture course, the notes tend to go into various (interesting!) digressions and cover additional material which is meant to provide the reader with a "larger and clearer picture". Some parts of the material which are additional and are not covered in the lectures are clearly labeled (as *deep dives*). However, this is not always possible so to know the examinable material you should attend the lectures. I should stress the examinable material is summarised in the syllabus and covered in the lectures – nothing less or more is examinable.

Thanks too to Jan, for the style file for the deep dives, and to Austin for the ducks!

These notes are work in progress and are being constantly improved. I am very grateful to all who have helped, or will help me to improve them.

At the moment there is a discussion forum on the course page for comments / corrections to the notes. Once the term is over, I'd appreciate further comments and corrections by email to `stuart.white@maths.ox.ac.uk`.

0.1 Overview / Background

Will follow - it's always best to write these last!

0.2 Notation

We will write $\mathbb{F}$ for either $\mathbb{R}$ or $\mathbb{C}$ when it does not matter which of these is the underlying field of our vector space. This course it will rarely matter, but when we focus on operators, particularly from B4.2 onwards it will often be advantageous to work over $\mathbb{C}$, as this is algebraically closed. You'll be familiar with the fact that complex square matrices always have eigenvalues, as the characteristic equation must have a solution over $\mathbb{C}$, but real square matrices need not have any real eigenvalues. This phenomena persists into the infinite dimensional setting: the spectrum of a bounded linear operator on a complex Banach space is always non-empty, and for this reason I prefer to work over the complex field when studying operators. When one is studying Banach spaces in their own right, typically one takes $\mathbb{F} = \mathbb{R}$ (to avoid sometimes needing to do arguments involving taking real parts, see for example the Hahn–Banach theorem), but this is far less important than the advantages we get from taking $\mathbb{F} = \mathbb{C}$ when we study operators between Banach spaces.

Deep Dive

Anything marked as a *Deep Dive* covers material outside of the syllabus. It is only intended for those who are interested and eager to understand things in more depth. It is non-examinable and not necessary for the course. It goes above and beyond the material, often indicating links with other courses and parts of mathematics.

Even the eager readers should skip those parts on the first reading. More deep dives may appear as I revise the notes. The depth of deep dives may vary considerably from one dive to another.

0.3 Course Synopsis

- Brief recall of material from Part A Metric Spaces and Part A Linear Algebra on real and complex normed vector spaces, their geometry and topology and simple examples of completeness.
- The norm associated with an inner product and its properties.
- Banach spaces, exemplified by ℓ^p, L^p and $C(K)$, and spaces of differentiable functions.
- Finite-dimensional normed spaces, including equivalence of norms and completeness.
- Hilbert spaces as a class of Banach spaces having special properties (illustrations, but no proofs); examples (Euclidean spaces, ℓ^2 and L^2), projection theorem, Riesz Representation Theorem.
- Density. Approximation of functions, Stone-Weierstrass Theorem. Separable spaces; separability of subspaces.
- Bounded linear operators, examples (including integral operators).
- Continuous linear functionals. Dual spaces.
- Statement of the Hahn-Banach Theorem; applications, including density of subspaces and embedding of a normed space into its second dual.
- Adjoint operators.

Contents

0 Introduction **1**
0.1 Overview / Background . . . 1
0.2 Notation . . . 1
0.3 Course Synopsis . . . 2

1 Normed spaces and Banach spaces **4**
1.1 Definitions and review from metric spaces . . . 4
1.2 Examples . . . 7
1.3 Constructions . . . 17

2 Inner product spaces and Hilbert spaces **19**
2.1 Definitions and basic properties . . . 19
2.2 Examples . . . 21
2.3 Orthogonality . . . 22
2.4 Orthonormal bases . . . 25

3 Bounded linear operators between normed vector spaces **27**
3.1 Boundedness and continuity . . . 27
3.2 Examples . . . 29
3.3 Properties of bounded linear operators . . . 33
3.4 Invertibility . . . 36

1 Normed spaces and Banach spaces

In this section we introduce Banach spaces as complete normed spaces. Our main objective is to give a range of examples. The important special case of Hilbert spaces when the norm is induced from an *inner product* is the subject of Section 2.

1.1 Definitions and review from metric spaces

The key tool linking used to perform analysis on vector spaces is a *norm*, a suitable notion of distance compatible with the linear algebra structure.

Definition 1.1. Let X be a vector space (over either $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$). A *norm* $\|\cdot\| : X \to \mathbb{R}$ is a function with the following properties:

(N1) for all $x \in X$, we have $\|x\| \geq 0$ with $\|x\| = 0 \Leftrightarrow x = 0$;

(N2) for all $x \in X$ and $\lambda \in \mathbb{F}$, we have $\|\lambda x\| = |\lambda|\|x\|$;

(N3) for all $x, y \in X$, the *triangle inequality* $\|x+y\| \leq \|x\| + \|y\|$ holds.

We call a pair $(X, \|\cdot\|)$ a *normed space*. We will often suppress explicit mention of the norm and say 'Let X be a normed space.'

Every norm $\|\cdot\|$ induces a metric
$$d : X \times X \to \mathbb{R}$$
via $d(x,y) := \|x-y\|$ and so all standard notions and properties of a metric space encountered in part A are applicable. Recall:

- Definition of convergence of a sequence (x_n) in X to $x \in X$:
$$x_n \to x \quad \Longleftrightarrow \quad d(x_n, x) \to 0 \quad \Longleftrightarrow \quad \|x_n - x\| \to 0.$$

- A sequence (x_n) in X is called *Cauchy* if:
$$\forall \varepsilon > 0,\ \exists N \text{ such that } \forall n, m \geq N,\ \|x_n - x_m\| < \varepsilon.$$

  This is written as $\|x_n - x_m\| \to 0$ as $m, n \to \infty$. Recall that a metric space X is called *complete* if and only if every Cauchy sequence in X converges to a point in X. For proving completeness it's worth reminding yourself of the following lemma. The proof, left as an exercise, is essentially the same as the last part of the proof of the that Cauchy sequences in $\mathbb{R}$ converge (using the Bolzano–Weierstrass theorem).

Lemma 1.2. *Let (x_n) be a Cauchy sequence in a normed space X. Then the following are equivalent:*

(i) *(x_n) converges,*

(ii) *(x_n) has a convergent subsequence.*

- A function $f : X \to Y$ is *continuous at* $x \in X$ if
$$\forall \varepsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x_0 \in X, \|x - x_0\| < \delta \implies \|f(x) - f(x_0)\| < \varepsilon.$$

The function f is *continuous* if it is continuous at x for all $x \in X$. These definitions, and in fact all properties of metric spaces,[1] have sequential characterisations: continuous functions are those that preserve limits of sequences. Precisely f is continuous at $x \in X$ if and only if for all sequences (x_n) in X with $x_n \to x$, we have $f(x_n) \to f(x)$. for every $x \in X$ and every sequence (x_n) in X

$$x_n \to x \implies f(x_n) \to f(x).$$

Setting this out in the context of normed spaces, $f : X \to Y$ is continuous if and only if whenever (x_n) is a sequence in X with $\|x_n - x\|_X \to 0$, we have $\|f(x_n) - f(x)\|_Y \to 0$.

- A set $U \subset X$ is open if for every $x_0 \in U$ there exists $r > 0$ such that

  $$B_r(x_0) := \{x \in X : \|x - x_0\| < r\} \subset U.$$

  Here $B_r(x_0)$ is the open ball centred at x_0 with radius r.

- By definition, a set $F \subset X$ is closed if its complement $X \setminus F$ is open. This can be characterised in terms of sequences: F is closed if and only if whenever (x_n) is a sequence of elements in F which converges to $x \in X$, then $x \in F$, i.e. F contains the limits of all convergent sequences.

The definition of the norm ensures that all the algebraic operations are automatically continuous:

- The scalar multiplication $\mathbb{F} \times X \to X$, $(\lambda, x) \mapsto \lambda x$ is continuous, where $\mathbb{F} \times X$ is given the product metric.
- The addition $X \times X \to X$, $(x, y) \mapsto x + y$ is continuous, where $X \times X$ is given the product metric (below we will give $X \times X$ a norm, but here we only need a metric).
- The norm $\|\cdot\| : X \to \mathbb{R}$ is continuous.

We also recall the appropriate notions of equivalence. First, the equivalence of different norms on the same space:

Definition 1.3. Let X be a vector space. Two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X are *equivalent* if and only if there exist a constant $C > 0$ so that for all $x \in X$

$$C^{-1}\|x\|_1 \le \|x\|_2 \le C\|x\|_1.$$

[1]The topology on a metric space is fully determined by knowledge of which sequences converge to which points. It is for this reason that any property which can be described using the metric can in some way be characterised by sequences. Therefore when I work with normed spaces, or more generally metric spaces, I have a tendancy towards giving sequence based arguments.

Deep Dive

Altthough this course, and Functional Analysis II do not rely on the Part A topology course, I can't resist pointing out that while one can't 'do everything with sequences' in a general topological space, there is a suitable generalisation of a sequence – known as a net – and one can do everything with nets. For example a function $f : X \to Y$ between topological spaces is continuous if and only if whenever $x_i \to x$ is a convergent net in X, then $f(x_i) \to f(x)$ is a convergent net in Y. I've found that the sort of functional analysis arguments I need to do in general topological spaces, work well with nets — sometimes all one needs to do is replace all the n's indexing the sequence with an i indexing a net! Of course you need to know about nets for this; the classic place which I learnt this from is Kelley's 'General topology', and another source is Willard's 'General Topology' (the section on nets is nice and short, but you'll need to dig around elsewher to learn that, for example, a subset is compact if and only if every net has a convergent subnet).

The normed spaces $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ are equivalent precisely when the metrics are strongly equivalent in the sense of the metric spaces course.[2] This ensures that all metric notions (open sets, convergent sequences, Cauchy sequences etc) are all the same in both norms.

Secondly, the notion of isomorphism between normed spaces. While there is only one possible notion of an isomorphism between vector spaces –

Definition 1.4. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Say X and Y are *isometrically isomorphic* if there exists a surjective linear map $T\colon X \to Y$ with $\|T(x)\|_Y = \|x\|_X$ for all $x \in X$.[3] We call T an *isometric isomorphism*. Say X and Y are *isomorphic* if there exists an isomorphism $T\colon X \to Y$ of vector spaces which is also a homeomorphism of the underlying metric spaces, i.e T and T^{-1} are continuous.

We will have much more to say about the linear operators T appearing here in Section **??** and beyond. For now it is useful to know that $(X, \|\cdot\|_1)$ and $(X, \|\cdot\|_2)$ can be (isometrically) isomorphic, without being equivalent (Problem sheet 1 asks you to produce an example).

Deep Dive

You've met a number of theorems in your analysis courses so far that give conditions under which a continuous bijection is a homeomorphism, i.e. has continuous inverse. For example in prelims, a continuous bijection between two intervals has continuous inverse, or a continuous bijection from a compact metric (or topological) space into a metric space (or Hausdorff topological space) has continuous inverse. In functional analysis we have the following consequence of Banach's open mapping theorem, proved in B4.2 using the baire category theorem: if $T\colon X \to Y$ is a bijective continuous linear map between *Banach* spaces, then T^{-1} is continuous, and hence X and Y are isomorphic Banach spaces. This is sometimes called the Banach isomorphism theorem, and decreases the work needed to obtain an isomorphism between Banach spaces.

One of the key objects we study in this course are Banach spaces and linear maps between such spaces.

Definition 1.5. A normed space $(X, \|\cdot\|)$ is a Banach space if it is complete, i.e. if every Cauchy sequence in X converges.

A normed space is complete if and only if absolute convergence of series implies convergence of series:

Proposition 1.6. *Let X be a normed space. Then the following are equivalent*

(i) X is a Banach space,

(ii) Absolute convergence of series implies convergence, i.e. for sequences (x_n) in X and the corresponding partial sums $s_n := \sum_{k=1}^n x_k$ we have

$$\sum_{i=1}^{\infty} \|x_n\| < \infty \quad \Rightarrow \quad s_n \text{ converges to some } s \in X.$$

Proof. (i) $\implies$ (ii): Suppose X is complete and $\sum_{n=1}^{\infty} \|x_n\| < \infty$. Then, for $s_n = \sum_{k=1}^n x_n$, the sequence (s_n) is Cauchy as for $m > n \geq N$

$$\|s_n - s_m\| = \|\sum_{k=n+1}^{m} x_k\| \leq \sum_{k=n+1}^{m} \|x_k\| \leq \sum_{k=N+1}^{\infty} \|x_k\| \to 0 \text{ as } N \to \infty.$$

[2] You might be suspicious of the terminolgy: normed spaces are called equivalent when their underlying metric structures are strongly equivalent. But while in the generality of metric spaces, equivalence of two metrics d_1 and d_2 on X – defined in Part A metric spaces to mean that the identity map Id_X is a homeomorphism – is not the same notion as strong equivalence, for normed spaces it is. This will all follow from equivalence between continuity and boundedness of linear maps in Section **??**.

[3] Such a map T is necessarily injective, so T is an isomorphism of vector spaces which is also an isometry.

As X is complete we thus obtain that s_n converges to some element $s \in X$.

(ii) $\implies$ (i): Suppose (ii) holds and let (x_n) be a Cauchy sequence. Select a subsequence x_{n_j} so that

$$\|x_{n_j} - x_{n_{j+1}}\| \leq 2^{-j},$$

where the existence of such a subsequence is ensured by the fact that (x_n) is Cauchy. Then $\sum_{j=1}^{\infty} \|x_{n_{j+1}} - x_{n_j}\| \leq 1 < \infty$ so (ii) ensures that $\sum_{j=1}^{\infty}(x_{n_{j+1}} - x_{n_j})$ converges. Hence $x_{n_k} = x_{n_1} + \sum_{j=1}^{k-1}(x_{n_{j+1}} - x_{n_j})$ converges, so (x_n) has a convergent subsequence and must thus, by Lemma 1.2, itself converge. □

1.2 Examples

$(\mathbb{R}^n, \|\cdot\|_p)$ and $(\mathbb{C}^n, \|\cdot\|_p)$, $1 \leq p \leq \infty$. Consider $\mathbb{R}^n$, or $\mathbb{C}^n$, equipped with

$$\|x\|_p := \Big(\sum_i |x_i|^p\Big)^{1/p} \text{ for } 1 \leq p < \infty$$

respectively

$$\|x\|_\infty := \sup_{i \in \{1,\dots,n\}} |x_i|.$$

One can show that these are all norms, with the challenging bit being the proof of the triangle inequality

$$\|x+y\|_p = \Big(\sum_i |x_i + y_i|^p\Big)^{1/p} \leq \|x\|_p + \|y\|_p,$$

which is is the finite dimensional version of Minkowski's inequality.

Warning. This inequality does not hold if we were to extend the definition of $\|\cdot\|_p$ to $0 < p < 1$, and hence the above expression does not give a norm on $\mathbb{R}^n$ if $p < 1$.

We will write ℓ_n^p for the normed space $(\mathbb{R}^n, \|\cdot\|_p)$. A useful property to deal with the p norms $1 \leq p \leq \infty$ (and their generalisations to sequence and functions spaces) is Hölder's inequality (which we proved in much more generality in the Integration course – see Proposition 1.12).

Lemma 1.7 (Hölder's inequality in $\mathbb{R}^n$)**.** *For $1 \leq p, q \leq \infty$ with*

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{1}$$

we have that for any $x, y \in \mathbb{C}^n$

$$\Big|\sum_{i=1}^n x_i y_i\Big| \leq \|x\|_p \|y\|_q.$$

In (1) we use the convention that $\frac{1}{p} = 0$ for $p = \infty$, and one often calls numbers $p, q \in [1, \infty]$ satisfying (1) *Hölder conjugate exponents*.

Remark. As you will show on Problem sheet 1, we have that for all $1 \leq p < \infty$

$$\|x\|_\infty \leq \|x\|_p \leq n^{1/p}\|x\|_\infty.$$

Hence the ∞-norm is equivalent to every p-norm and thus, by transitivity, we have that $\|\cdot\|_p$ is equivalent to $\|\cdot\|_q$ for every $1 \leq p, q \leq \infty$. In fact as we will show in Section **??**, all norms on finite dimensional spaces are equivalent.

Deep Dive

With some small exceptions (which I neglected to mention in lectures), the spaces ℓ_n^p are not isometrically isomorphic as p varies. The first exception is if $n=1$: then all the norms are the same! The only other exception is if $n=2$, when ℓ_2^1 is isometrically isomorphic to ℓ_2^∞, via the map $(x,y)\to(x+y,x-y)$. In a deep dive below we discuss to how see this sort of thing for the sequence spaces ℓ^p, and come back to ℓ_n^1 and ℓ_n^∞ for $n\geq 3$.

Sequence spaces ℓ^p and c_0. An infinite dimensional analogue of $(\mathbb{R}^n,\|\cdot\|_p)$, respectively $(\mathbb{C}^n,\|\cdot\|_p)$ are the spaces of sequences ℓ^p, $1\leq p\leq\infty$, where for $1\leq p<\infty$

$$\ell^p := \Big\{(x_j)_{j\in\mathbb{N}} : \sum_{j=1}^\infty |x_j|^p<\infty\Big\}$$

equipped with $\|\cdot\|_p$ where for $1\leq p<\infty$

$$\|x\|_{\ell^p} = \Big(\sum_{j=1}^\infty |x_j|^p\Big)^{1/p},$$

while ℓ^∞ denotes the space of bounded sequences, equipped with

$$\|(x_j)\|_\infty := \sup_j |x_j|.$$

For any $1\leq j\leq\infty$ we have that ℓ^p is a normed space (where we define addition and scalar-multiplication component-wise). Again the main difficulty is to obtain Minkowski's inequality, which is precisely the triangle inequality.

Example 1.8. ℓ^p is complete for $1\leq p\leq\ \infty$.

This follows the standard procedure for showing completeness. Given a Cauchy sequence (x_n) in a normed space X:

1. identify a candidate x for $\lim x_n$;
2. show that $x\in X$;
3. show $\|x-x_n\|_X\to 0$ as $n\to\infty$.

Often, as in the proof below, which we recall from metric spaces, steps 2 and 3 can be performed simultaneously. The $p=\infty$ case is easier (and a special case of the Banach space $\mathscr{F}_b(\Omega)$ below, by taking $\Omega=\mathbb{N}$.)

Proof for $1\leq p<\infty$. Let $(x^{(n)})$, be a Cauchy-sequence in ℓ^p and write $x^{(n)}=(x_j^{(n)})_{j=1^\infty}$. As for every $j\in\mathbb{N}$

$$|x_j^{(n)}-x_j^{(m)}|\leq\|x^{(n)}-x^{(m)}\|_2\to 0$$

as $m,n\to\infty$, the sequence $(x_j^{(n)})$ is Cauchy in $\mathbb{F}$ so converges, say $x_j^{(n)}\to x_j$.

Fix $\varepsilon>0$. Then there exists N so that for all $n,m\geq N$

$$\|x^{(n)}-x^{(m)}\|_p\leq\varepsilon.$$

Thus for every $K \in \mathbb{N}$ and for all $n \geq N$ we have that

$$\sum_{j=1}^{K} |x_j^{(n)} - x_j|^p = \lim_{m\to\infty} \sum_{j=1}^{K} |x_j^{(n)} - x_j^{(m)}|^p \leq \varepsilon^p.$$

Letting $K \to \infty$, it follows that for all $n \geq N$, we have $x^{(n)} - x \in \ell^p$, and so $x \in \ell^p$ (step 2), and $\|x^{(n)} - x\|_p \leq \varepsilon$ for $n \geq N$,[4], though for sums we were able to d so that $x^{(n)} \to x$ in ℓ^p. □

Again Hölder's inequality is valid. That is for every $1 \leq p, q \leq \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$ and any $(x_j) \in \ell^p$ and $(y_j) \in \ell^q$ we have that $\sum x_j y_j$ converges, i.e. the pointwise product $xy \in \ell^1$ and

$$|\sum_j x_j y_j| \leq \|(x_j)\|_p \|(y_j)\|_q.$$

Again this is a special case of Proposition 1.12 from Integration, taking the measure space to be $\mathbb{N}$ equipped with counting measure.

We will sometimes also consider the subspace

$$c_0 := \{(x_n) \in \ell^\infty : x_n \to 0\}$$

of ℓ^∞, which is closed[5] and hence, when equipped with the ℓ^∞- norm a Banach space (see Proposition 1.16 below)

Note that all of these sequence spaces contain a common subspace c_{00} – the collection of sequences which are eventually zero. You should check that c_{00} is dense in c_0, and in ℓ^p for $1 \leq p < \infty$. Accordingly it can not be a Banach space in any of the p-norms (by Proposition 1.16 below).

Deep Dive

In fact there is no norm on c_{00} under which it is a Banach space.

Theorem. *No infinite dimensional Banach space X can have a countable* Hamel basis.

A *Hamel basis* is what up to now we've just called a basis, i.e. a linearly independent spanning set. This is a purely algebraic notion so $\mathscr{S}$ is a Hamel basis for X when no non-trivial *finite* linear combination of elements of $\mathscr{S}$ can be zero, and every element of X can be written as a finite linear combination of elements of $\mathscr{S}$. While in a normed space we are allowed to consider infinite sums, these are not used to define Hamel bases. Certainly the polynomials have a countably infinite Hamel basis and so can not be a Banach space under any norm.

For background, by a Zorn's lemma argument every vector space has a Hamel basis. Zorn's lemma is a tool equivalent to the axiom of choice, which will appear in some other deep dives, but is definitely non-examinable. Zorn's lemma will be described in B1.2 (Set Theory) which will show that every vector space having a Hamel basis is another reformulation of the axiom of choice (that course will use the language of basis, rather than Hamel basis, as the vector spaces there do not come equipped with norms).

The normal, and in my view best way, to prove that no Banach space can have a countably infinite Hamel basis is through the Baire category theorem, which will be proved in B4.2 Functional Analysis 2. This states that a complete metric space is never a countable union of nowhere dense subsets.[a] We will see in Section **??** that finite dimensional subspaces of normed spaces are always closed, and it is an easy exercise to check that a proper closed subspace of a normed space is nowhere dense. Then the result follows. On problem sheet 2

[4] We have done a by-hand version of Fatou's lemma for infinite sums here. I make this remark only so you can compare it with the use of Fatou's lemma in one of the proofs that L^p is complete.

[5] This is very similar to Problem sheet 1, B.1(a).

we will see an alternative, slightly messier proof in Section C, using Riesz's lemma from Section **??**.

This result shows that Hamel bases, while they generally exist, are unlikely to be all that useful for working with Banach spaces. Even for spaces such as ℓ^1 or c_0, where we clearly only need countably many bits of data to specify elements, do not have countable bases. Instead, one needs the appropriate analytic notion of a basis – *Schauder bases* – to develop a satisfactory theory for Banach spaces. These appear in more detail at the end of the C4.1 course, but they are perfectly accessible to you now. A good book taking a basis first approach, and so developing interesting properties of the sequence spaces is Carothers 'A short course in Banach space theory'. Definitely the first 6 chapters of this book are very readable along side this course - but from chapter 3 onwards go in different direction.

[a] A nowhere dense set is a set whose closure has empty interior.

When I'm trying to build counter examples, my first thought is to check the finite dimensional case, and assuming that doesn't work then I tend to look for an example using sequences.

Deep Dive

While for $n \in \mathbb{N}$ fixed, all the n-dimensional normed spaces ℓ_n^p are equivalent, this fails in infinite dimensions. Also, just as in the finite dimensional case, by looking at the geometry of the unit balls one can see that no pair of these spaces is isometrically isomorphic. One thing you can look at here is the *modulus of convexity*, which quantifies the convexity of the unit ball. For a Banach space X, define the modulus of convexity $\delta_X : [0,2] \to [0,1]$ by

$$\delta_X(\varepsilon) = \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : x, y \in X,\ \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\right\}.$$

This measures how far the average of two points x and y of distance at least ε on the sphere must get pushed inside the unit ball. (There's some nice pictures to be drawn here; if someone draws them I'll be happy to include). You should be able to see that for ℓ^∞ and ℓ^1 we have $\delta = 0$. The modulus of convexity of ℓ^2, and indeed any Hilbert space, is $\delta(\varepsilon) = 1 - \sqrt{1-\varepsilon^2/4} = \varepsilon^2/8 + O(\varepsilon^4)$ as $\varepsilon \to 0$.footnoteIt's small ε we care about, otherwise we wouldn't have used the notation ε. A two-dimensional geometric argument shows that this is the upper bound of the modulus of convexity for any normed space X; in this sense Hilbert space's have the "convexest" unit ball. For other p, we have

$$\delta_{\ell^p}(\varepsilon) = \begin{cases} (p-1)\frac{\varepsilon^2}{8} + o(\varepsilon^2), & 1 < p \leq 2; \\ \frac{\varepsilon^p}{p2^p}, & 2 \leq p < \infty. \end{cases}$$

While the modulus of convexity doesn't distinguish ℓ_n^∞ and ℓ_n^∞, for $n \geq 3$, you can see that nevertheless they are not isometrically isomorphic from the geometry of the unit balls: the ball of ℓ_n^1 has $2n$ extreme points, and the ball of ℓ_n^∞ has 2^n extreme points.[a]

Showing that in infinite dimensions none of the sequence spaces ℓ^p or c_0 are isomorphic is more challenging. By section **??** we will know that ℓ^∞ is not isomorphic to any of the other spaces — it is too big (not separable to be precise). By the end of the course, we will know that c_0, ℓ^1 are not isomorphic and also not isomorphic to any other ℓ^p space. In fact all these spaces are pairwise non-isomorphic. This is much harder, and we won't prove it in any of the functional analysis courses here. (The way this is normally done is through Pitt's theorem, that says that for $1 \leq p < q < \infty$, any bounded linear map (see Section 3) $T : \ell^q \to \ell^p$ is compact (a concept that will be defined in B4.2). Taking this fact for granted (a proof can be found in

Chapter 2 of Albaic and Kalton's "Topics in Banach Space Theory") you should be able to deduce that all these sequence spaces are pairwise non-isomorphic by the end of B4.2.

[a] I'll let you formalise or look up the definition of an extreme point. A picture ought to work in finite dimensions, but in infinite dimensions some unit balls need not have extreme points, for example c_0. I will make further deep dive remarks on this later in the notes.

Function spaces with supremum-norm

The supremum norm is often used to make spaces of bounded functions complete.

Example 1.9. Let Ω be any set. Then

$$\mathscr{F}^b(\Omega) := \{f : \Omega \to \mathbb{F} \text{ bounded}\}$$

is a Banach space under the norm $\|f\|_{\sup} = \sup_{x\in\Omega} |f(x)|$. (We will often drop the subscript sup on these norms).

Proof of completeness of $\mathscr{F}^b(\Omega)$. This is another example of the three step process. If (f_n) is Cauchy in $\mathscr{F}^b(\Omega)$ then for each $x \in \Omega$, $(f_n(x))$ is Cauchy so converges to $f(x) \in \mathbb{F}$, say. Fixing $\varepsilon > 0$, there is $N \in \mathbb{N}$ with $|f^{(n)}(x) - f^{(m)}(x)| \leq \varepsilon$ for $m, n \geq N$ and all $x \in \Omega$. Taking limits as $m \to \infty$, $|f_n(x) - f(x)| \leq \varepsilon$ for all $x \in \Omega$ and $n \geq N$. Therefore f is bounded, and $\|f_n - f\| \leq \varepsilon$. Accordingly $\mathscr{F}^b(\Omega)$ is complete. □

For a metric[6] space K, the vector space $C_b(K) := \{f : K \to \mathbb{F} \text{ continuous and bounded}\}$ is closed in $\mathscr{F}^b(K)$,[7] and hence $C_b(K)$ is a Banach space with the sup norm (see Proposition 1.16). If K is compact, then all continuous functions are bounded, and we write $C(\Omega) := \{f : \Omega \to \mathbb{F} \text{ continuous}\} = C_b(\Omega)$.

Example 1.10. Let K be a compact metric space. Then $C(K)$ is a Banach space with the sup-norm.

Similarly, on spaces of differentiable functions (with bounded derivatives) such as $C^1([0,1])$ – the space of functions on $[0,1]$ which have continuous derivatives (including at the end points), to get completeness we will need norms that are built using the sup norm of both the function and its derivative. Indeed, since $C^1([0,1])$ is dense in $C[0,1]$ (in fact the smaller set of polynomials is dense in $C[0,1]$ – see Section **??**), it is not closed in the sup-norm and so is not a Banach space with respect to $\|\cdot\|_{\sup}$. Instead we use a standard trick for creating a norm on spaces like this; take the sum of two norms that we want to be able to control. Then a sequence which is Cauchy in the sum of the norms, will necessarily be Cauchy in each norm.

Example 1.11. $C^1([0,1])$ is a Banach space with $\|f\|_{C^1} := \|f\|_{\sup} + \|f'\|_{\sup}$

Proof. Suppose (f_n) is a Cauchy sequence in $C^{(}[0,1])$. Then (f_n) is $\|\cdot\|_{\sup}$-Cauchy so converges to some $f \in C([0,1])$ and (f_n') is also $\|\cdot\|_{\sup}$-Cauchy so converges to some $g \in C([0,1])$. Now prelims comes to the rescue: since (f_n) converges uniformly to f and each f_n' is differentiable and f_n' converges uniformly to g, it follows that f is differentiable and $f' = g$. Hence $f \in C^{(}[0,1])$, and then $\|f_n - f\|_{C^1} \to 0$. □

Deep Dive

The map $f \mapsto (f', f(0))$ gives an isomorphism[a] from $C^1([0,1])$ to $C([0,1]) \times \mathbb{R}$ (we will discuss norms on the Cartesian product below. In fact, using a Schauder basis for $C([0,1])$ it is possible to show that $C([0,1]) \cong C([0,1]) \times \mathbb{R}$ though the isomorphism can not be isometric,[b] so that as Banach spaces $C^1([0,1])$ and $C([0,1])$ are isomorphic. But in applications we would most likely care about how our elements are

[6] this works fine for a topological space

[7] this is a consequence of the fact that a uniform limit of continuous functions is continuous — a 3ε or $\varepsilon/3$ argument depending on your taste.

realised as functions: so while these Banach spaces are abstractly isomorphic, it makes sense to understand them separately.

[a]This map is bounded and bijective between Banach spaces, so an isomorphism from Banach's isomorphism theorem; though you could show directly that the inverse map is bounded.

[b]We're implicitly working with reals here, so you can use the geometry of the unit ball: $C([0,1])$ has two extreme points, the constant functions ± 1, while the unit ball of $C([0,1]\times\mathbb{R}$ with the ℓ^2-norm has more.

Function spaces L^p, $1 \le p \le \infty$

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space (considering examples like Ω an interval in $\mathbb{R}$ or a measurable subset of $\mathbb{R}^n$ with Lebesgue measure will be more than sufficient for the course). Consider for $1 \le p < \infty$ the space of functions

$$\mathscr{L}^p(\Omega) := \Big\{ f : \Omega \to \mathbb{R} \text{ (or } \mathbb{C}) \text{ measurable so that } \int_\Omega |f|^p dx < \infty \Big\}$$

respectively

$$\mathscr{L}^\infty := \Big\{ f : \Omega \to \mathbb{R} \text{ (or } \mathbb{C}) \text{ measurable so that } \exists M \text{ with } |f| \le M \text{ a.e.} \Big\}.$$

This is not a course on Lebesgue integration, so typically in examples we will only work with measurable functions; you may assume that all functions you encounter are measurable when needed (though don't let that stop you briefly reminding yourself why). But of course not all measurable functions are integrable and that indeed for a general measurable function the integral might not even be defined, so justification is needed to consider integrals in general. However we also recall that the integral of a non-negative functions f is always defined though might be infinite.

We equip these spaces with

$$\|f\|_p := \Big(\int_\Omega |f|^p dx \Big)^{1/p} \text{ for } 1 \le p < \infty$$

respectively

$$\|f\|_\infty := \text{ess sup}|f| := \inf\{M > 0 : |f| \le M \text{ a.e. } \}.$$

We note that $\|\cdot\|_p$ is only a seminorm on $\mathscr{L}^p$ with $\|f-g\|_p = 0$ if and only if $f = g$ a.e. We can hence turn $(\mathscr{L}^p, \|\cdot\|)$ into a normed space by taking the quotient with respect to the equivalence relation

$$f \sim g \Longleftrightarrow f = g \text{ a.e.}.$$

The resulting quotient space

$$L^p(\Omega) := \mathscr{L}^p / \sim \text{ equipped with } \|\cdot\|_p$$

is one of the most important spaces of functions in the modern theory of PDE, and will be further developed in the course C4.3 Functional analytic methods for PDEs. Recall the following two key inequalities for L^p spaces; the first giving the triangle inequality, and the second will be crucial later in the course when we examine the dual spaces of $L^p(\Omega)$.

Proposition 1.12 (Minkowski and Hölder).

- *The Minkowski-inequality, which is the triangle inequality for L^p holds: for $f, g \in L^p(\Omega)$, we have*

$$\|f+g\|_p \le \|f\|_p + \|g\|_p.$$

- *Hölder's inequality holds: If $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ where $\frac{1}{p} + \frac{1}{q} = 1$ then their product fg is integrable with*

$$\Big| \int_\Omega fg dx \Big| \le \|f\|_p \|g\|_q.$$

None of the L^p norms are equivalent, though when Ω has positive and finite measure, we can estimate the L^p norm of functions by their L^q norm if $p < q$ and we have

$$L^\infty(\Omega) \subsetneq L^q(\Omega) \subsetneq L^p(\Omega) \subsetneq L^1(\Omega) \text{ for any } 1 < p < q < \infty. \tag{2}$$

As an example consider $\Omega = (0,2) \subset \mathbb{R}$ and $p = 2$, $q = 4$. Adding in a multiplication by the constant function $g = 1$ we can estimate, using Hölder's inequality,

$$\|f\|_{L^2}^2 = \int |f|^2 \cdot 1 dx \le \||f|^2\|_{L^2}\|1\|_{L^2} = \Big(\int_0^2 f^4\,dx\Big)^{1/2} \cdot \Big(\int_0^2 1\,dx\Big)^{1/2} = \sqrt{2}\|f\|_{L^4}^2,$$

so we get $\|f\|_{L^2} \le \sqrt{2}\|f\|_{L^4}$ and in particular that every $f \in L^4([0,2])$ is also an element of $L^2([0,2])$. The general case is discussed on the first problem sheet.

Warning.
- Note that the inclusions of the function spaces $L^p(\Omega)$ for sets Ω with bounded measure are the "other way around" compared with the inclusions of the sequence spaces ℓ^p.
- The inclusion (2) is wrong for unbounded domains, e.g. the constant function $f = 1$ is an element of $L^\infty(\mathbb{R})$ but isn't contained in any $L^p(\mathbb{R})$, $1 \le p < \infty$.

Remark. In practice it is can be useful to extend $\|\cdot\|_{L^p}$ to a function from the space of all (measurable) functions to $[0,\infty) \cup \{\infty\}$ by simply setting $\|f\|_{L^p} = \infty$ if $\int |f|^p = \infty$ (respectively for $p = \infty$ if $f \notin L^\infty$), and we note that also with this 'abuse of notation' the triangle and Hölder-inequality still hold (with the convention that $0 \cdot \infty = 0$ for Hölder's inequality). Similarly we can extend $\|\cdot\|_p$ to a function that maps all sequences to $[0,\infty) \cup \{\infty\}$ but we stress that while this notation/convention can be useful and used in the literature, these functions into $[0,\infty) \cup \{\infty\}$ are not norms as a norm is by definition a function into $[0,\infty)$.

Example 1.13. $(L^\infty(\Omega), \|\cdot\|_{L^\infty})$ is a Banach space.

The proof is more or less similar to the proof of completeness for $\mathscr{F}_b(\Omega)$, or a direct proof of completeness for $C(K)$ in the supremum norm, except that we have to take care of the almost everywhere nature of things.

Proof. Let (f_n) be a Cauchy sequence in $L^\infty(\Omega, \mathbb{R})$. For each $k \in \mathbb{N}$, there exists N_k such that

$$\|f_n - f_m\|_{L^\infty} \le \frac{1}{k} \text{ for all } n, m \ge N_k.$$

This means that, for each k and $m, n \ge N_k$ there is a null subset $Z_{k,m,n}$ of Ω such that

$$|f_n(x) - f_m(x)| \le \frac{1}{k} \text{ for } x \in \Omega \setminus Z_{k,m,n}.$$

Let $Z = \cup_k \cup_{n,m \ge N_k} Z_{k,n,m}$, which, as a countable union of null set, is null. Then,

$$|f_n(x) - f_m(x)| \le \frac{1}{k} \text{ for all } n, m \ge N, x \in \Omega \setminus Z. \tag{3}$$

So for almost all $x \in \Omega$, $(f_n(x))$ is Cauchy, and hence converges to some $f(x)$.

Being an almost everywhere limit of measurable functions, f is measurable. Sending $m \to \infty$ while keeping n fixed in (3) we get

$$|f_n(x) - f(x)| \le \frac{1}{k} \text{ for all } n \ge N_k, x \in \Omega \setminus Z.$$

This shows that $\|f_n - f\|_{L^\infty} \le \frac{1}{k}$ for all $n \ge N_k$. This implies on one hand that $f_n - f$ and hence f belong to $L^\infty(\Omega)$ and on the other hand that $f_n \to f$ in $L^\infty(\Omega)$. $\square$

Example 1.14. L^p is complete for $1 \le p < \infty$.

We give two proofs. Firstly we consider the proof lectured in part A integration,[8] and show how this really fits into the abstract framework of proving completeness by showing that absolutely convergence series converge.

Proof. Let $(f_n)_{n=1}^\infty$ be a sequence in $L^p(\Omega)$ with $\sum_n \|f_n\|_p < \infty$. Define $g_n = \sum_{r=1}^n |f_r|$. This gives an increasing sequence of non-negative measurable functions which converges to $g = \sum_{n=1}^\infty |f_n|$ (g can of course take the value ∞ whenever this sum diverges). By Minkowski

$$\int g_n^p = \|g_n\|_p^p \le \left(\sum_{r=1}^n \|f_n\|_p\right)^p \le \left(\sum_{n=1}^\infty \|f_n\|_p\right)^p < \infty.$$

Therefore by the monotone convergence theorem g^p is integrable, i.e. $g \in L^p$, and hence g is finite almost everywhere. That is the sum $\sum_{n=1}^\infty f_n$ is absolutely convergent almost everywhere, and so converges almost everywhere, say to f.[9] Moreover, applying the triangle inequality pointwise $|f|^p \le g^p$, so $f \in L^p$ by comparison.[10]

Finally, another application of the triangle inequality gives

$$|f - \sum_{r=1}^n f_r|^p \le \left(\sum_{r=n+1}^\infty |f_r|\right)^p \le g^p,$$

so the dominated convergence theorem gives $\|f - \sum_{r=1}^n f_r\|_p^p \to 0$, and $\sum_{n=1}^\infty f_n = f$ in L^p.[11] □

In the integration course proved the following facts (the first of which essentially came from in the middle of the proof given there that L^p was complete which essentially used the series argument above):

- If (f_n) is a Cauchy sequence in L^p (or a sequence converging in L^p to $f \in L^p$), then there exists a subsequence f_{n_k} which converges almost everywhere (to f). But we can not guarentee that f_n converges almost everwhere, only that a subsequence does.
- Given a sequence (f_n) in L^p with $f_n \to f$ almost everywhere, the *convergence theorems* (monotone convergence theorem, Fatou's lemma and the dominated convergence theorem) give tools you can try and use to prove that $f \in L^p$ and $f_n \to f$ in L^p.

While we only really used Fatou's lemma in the integration course as a tool for obtaining the dominated converge theorem, it can be very useful for obtaining convergence of f_n to f in the L^p norms. Let's see this in action in our second proof of completeness of L^p assuming the first fact above.

Second proof of completeness of L^p. Let $(f_n)_{n=1}^\infty$ be Cauchy in L^p, and let (f_{n_k}) be a subsequence which converges almost everywhere to a (necessarily measurable) f. Fix $\varepsilon > 0$. We know that

$$\|f_n - f_{n_j}\|_{L^p}^p = \int_\Omega |f_n - f_{n_j}|^p \, dx \le \varepsilon^p \text{ for all } n, n_j \ge N.$$

As $j \to \infty$, the a.e. limit of the integrand is $|f_n - f|^p$. Moreover, the integrand is non-negative. By Fatou's lemma,[12] we have

$$\int_\Omega |f_n - f|^p \, dx \le \liminf_{j\to\infty} \int_\Omega |f_n - f_{n_j}|^p \, dx \le \varepsilon^p \text{ for all } n \ge N.$$

In other words, $\|f_n - f\|_{L^p} \le \varepsilon$ for all $n \ge N$. This implies on one hand that $f_n - f$ and hence f belong to $L^p(\Omega)$ and on the other hand that $f_n \to f$ in $L^p(\Omega)$. □

[8] The case of L^p was lectured, but only the case of L^1 is in the lecture notes.

[9] This is Step 1 of the process for completeness (in this absolute convergence framework) by providing a candidate limit.

[10] Step 2: the limit is in the space it should be in.

[11] The expression in L^p relating to this sum means that we have justified convergence of the sum in the norm on L^p, as required for Step 3.

[12] Compare this with footnote [4].

Since the first fact above is fair game for the course – we proved it in integration – this is an approach I encourage you to keep in mind.

Deep Dive

Let's give another proof that every L^p Cauchy-sequence has a subsequence which is almost everywhere convergent by means of *convergence in measure*, and Borel-Cantelli arguments (see B8.1, or many past integration exams). I don't think we'll really need these tools in the course, but you are welcome to use them in L^p type examples if they help.

Say that a sequence f_n of measurable functions on $(\Omega, \mathscr{F}, \mu)$ *converges in measure* to a measurable function f when for all $\delta > 0$,

$$\mu(\{x \in \Omega : |f_n(x) - f(x)| > \delta\}) \to 0 \text{ as } n \to \infty.$$

Likewise, (f_n) is Cauchy in measure if for all $\delta > 0$,

$$\mu(\{x \in \Omega : |f_n(x) - f_m(x)| > \delta\}) \to 0 \text{ as } n, m \to \infty.$$

Convergence in measure, implies being Cauchy in measure. For the converse we find an almost everywhere convergent subsequence to obtain the proposed limit.

Lemma. Suppose (f_n) is Cauchy in measure. Then there exists a subsequence (f_{n_k}) which is convergent almost everywhere to f, and f_n converges to f in measure.

Proof. To follow. □

Combining the above with the following lemma, every L^p Cauchy sequence has a subsequence which is convergent almost everywhere.

Lemma. Let (f_n) be Cauchy in $L^p(\Omega, \mathscr{F}, \mu)$. Then (f_n) is Cauchy in measure.

Proof. Fix $\varepsilon > 0$ and $\delta > 0$ and find N such that for $m, n \geq N$, we have $\|f_n - f_m\|_p < \varepsilon$. Therefore, for $n, m \geq N$,

$$\mu(\{x : |f_n(x) - f_m(x)|^{\geq} \delta^p\}) \leq \frac{\varepsilon^p}{\delta^p}.$$

Therefore (f_n) is Cauchy in measure. □

Deep Dive

We should also discuss potential isomorphisms between all the L^p-spaces and all the ℓ^p spaces. Firstly none of the L^p spaces are isometrically isomorphic; the same modulus of convexity formula in one of the deep dives above works for L^p.

$L^2(\Omega, \mathscr{F}, \mu)$ is a Hilbert space, and as we will learn these are determined up to isometric isomorphism by the size of an orthonormal basis. In most examples of interest to us, $(\Omega, \mathscr{F}, \mu)$ has just the right 'size' to be separable (have a countable dense subset; see Section **??**), and in this case it will be isometrically isomorphic to ℓ^2. This will always be the case when $\Omega \subset \mathbb{R}^n$ is a Lebesgue measurable subset (with non zero measure) equipped with Lebesgue measure.[a] As an example you might well be able to guess an isometric isomorphism between $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$.

A surprising theorem of Pełczyński shows that $L^\infty([0,1])$ and ℓ^∞ are isomorphic Banach spaces (though they are not isometrically isomorphic). This theorem proceeds through a back and forward principle known as Pełczyński's decomposition technique (see Albiac and Kalton, Theorem 2.3.3): if X and Y are Banach spaces such that X is isomorphic to a complemented subspace of Y[b], Y is isomorphic to a complemented subspace of X, X is isomorphic to $X \oplus X$, and Y is isomorphic to $Y \oplus Y$, then X and Y are isomorphic. Let $X = L^\infty([0,1])$ and $Y = \ell^\infty$. In fact both these spaces are what's known as *injective* (something we will find easier to define once we have bounded linear maps) which means that whenever either X or Y is isomorphic to a closed subspace of another Banach space, then this subspace is automatically complemented. We will be able to prove injectivity of ℓ^∞ as a consequence of Hahn–Banach by the end of the course. Injectivity of $L^\infty([0,1])$ is a bit harder; this can be found in Section 4.3 of Albiac and Kalton. Given these results to prove Pełczyński's theorem you need to see how to find ℓ^∞ as a closed subspace of L^∞ (have a go – it's not so bad), and L^∞ as a closed subspace of ℓ^∞ (this can be found as an extension exercise to Sheet 3). All of this is collected as Theorem 4.3.10 of Albiac and Kalton. To the best of my knowledge, Pełczyński's theorem is non-constructive and no explicit isomorphism is known.

Finally, ℓ^p and $L^p([0,1])$ are not isomorphic for other values of p. When $p = 1$ you'll be able to use something called Schur's property to distinguish ℓ^1 and $L^1([0,1])$ as an exercise in B4.2 and C4.1 (though I think the name 'Schur's property' will only be introduced in C4.1). To distinguish ℓ^p and $L^p([0,1])$ for $1 < p < 2$ and $2 < p < \infty$ one can show that ℓ^2 is a complemented subspace of $L^p([0,1])$ for any $1 < p < \infty$ (see Proposition 6.4.2 of Albiac and Kalton) but (by means of another Pełczyński decompositon technique) any complemented infinite dimensional subspace of ℓ^p is isomorphic to ℓ^p (see Theorem 2.2.4 of Albiac and Kalton).

[a]In general you need that Ω is σ-finite and $\mathscr{F}$ is the completion of a countably generated σ-algebra.

[b]we will discuss complemented subspaces in Section **??**

Some incomplete spaces

Example 1.15. We can construct many examples of non-complete spaces by equipping a well known space such as C_b, C^1, ℓ^p, L^p with the 'wrong' norm, or by choosing a subspace of a Banach space that is not closed. As an example we show that $C([0,1])$ equipped with $\|f\|_{L^1} = \int_0^1 |f| dx$ is not complete (which is actually an exercise from the metric spaces course and so I won't lecture it).

Proof. We give three proofs: one by direct argument, one via Corollary 1.6, and finally through density.

1. Let
$$g_n(x) = \begin{cases} (2x)^n & \text{for } x \in [0,1/2), \\ 1 & \text{for } x \in [1/2,1]. \end{cases}$$
For $n < m$, we have
$$\|g_n - g_m\|_{L^1} = \int_0^{1/2} [(2x)^n - (2x)^m]\, dx = \frac{1}{2(n+1)} - \frac{1}{2(m+1)},$$
so (g_n) is Cauchy. On the other hand, (g_n) is a decreasing sequence of non-negative functions which is bounded from above by 1. Its pointwise limit is the characteristic function of the interval $[1/2,1]$. By Lebesgue's dominated convergence theorem, g_n converges to $\chi_{[1/2,1]}$ in L^1 and there is no continuous function which is equal to $\chi_{[1/2,1]}$ almost everywhere,[13], and hence not in $C([0,1])$. In other words (g_n)

[13] $\chi_{[1/2,1]}$ is almost everywhere continuous, but not almost everywhere equal to a continuous function

has no limit in $C([0,1])$.[14]

2. For
$$f_n(x) := \begin{cases} 1 - n^2 x & \text{for } x \in [0, \frac{1}{n^2}] \\ 0 & \text{else} \end{cases}$$
we have that $\|f_n\|_1 = \frac{1}{2n^2}$ so $\sum \|f_n\|_{L^1}$ converges. However $\sum f_n$ cannot converge to an element of $C([0,1])$. Indeed suppose, seeking a contradiction, that $\sum f_n \to f$ converges in L^1 to a function $f \in C([0,1])$. Then, as continuous functions on compact sets are bounded, there exists some $M \in \mathbb{R}$ so that $f \leq M$ on $[0,1]$. Hence choosing $N \in \mathbb{N}$ so that $N \geq 2(M+1)$ we obtain that for any $n \geq N$ and any $x \in [0, \frac{1}{2N^2}]$
$$\sum_{j=1}^{n} f_j(x) - f(x) \geq \sum_{j=1}^{N} \frac{1}{2} - f(x) \geq N/2 - M \geq 1$$
and thus in particular $\|\sum_{j=1}^{n} f_j - f\|_1 \geq \frac{1}{2N^2} \not\to 0$.

3. We know from part A integration that $C([0,1])$ is a proper dense subspace of $L^1([0,1])$, so can not be complete (by Proposition 1.16).

□

1.3 Constructions

We end this section with a brief collection of ways to construct new normed spaces from existing examples, and when this preserves completeness.

Subspaces We first note that for any given subspace Y of a normed space $(X, \|\cdot\|)$ we obtain a norm on Y simply by restricting the given norm to Y. For the resulting normed space $(Y, \|\cdot\|)$ we have

Proposition 1.16. *Let X be a Banach space, $Y \subset X$ a subspace. Then*
$$(Y, \|\cdot\|) \textit{ is complete } \Leftrightarrow Y \subset X \textit{ is closed .}$$

Proof. Suppose Y is complete, and (y_n) is a sequence in Y with $y_n \to x \in X$. As (y_n) converges in X, it is Cauchy. Therefore by completeness it converges to some $y \in Y$. Hence $x = y \in Y$ by uniqueness of limits and Y is closed.

Conversely, suppose Y is closed in X and let (y_n) be a Cauchy sequence in Y. By completeness of X, it follows that there exists $x \in X$ with $y_n \to x \in X$. But as Y is closed we must have that $x \in Y$ and hence that (y_n) converges in Y. Therefore Y is complete. □

Direct sums Given two normed spaces X and Y we can define a norm on $X \times Y$ e.g. by
$$\|(x,y)\|_2 = (\|x\|^2 + \|y\|^2)^{1/2} \tag{4}$$
or more generally using any of the p-norms on $\mathbb{R}^2$ to define
$$\|(x,y)\|_p := \|(\|x\|, \|y\|)\|_p = (\|x\|^p + \|y\|^p)^{1/p} \text{ respectively } \|(x,y)\|_\infty := \max(\|x\|, \|y\|)$$

[14] Here we are using the fact that we know $C([0,1])$ is a subspace of $L^1([0,1])$ so limits are unique. When you did this exercise in the metric spaces course, this answer would not have been sufficient as we didn't have the space $L^1([0,1])$ to work with. In part A metric spaces you where supposed to deal with this by showing barehands that there is no continuous function which can arise as the L^1 limit.

where here and in the following we simply write $\|\cdot\|$ instead of $\|\cdot\|_X$ and $\|\cdot\|_Y$ if it is clear from the context what norm we are using. As all (the ℓ^p)-norms on $\mathbb{R}^2$ are equivalent, it follows that all the norms $\|(x,y)\|_p$ are equivalent on $X \times Y$. We tend to write $X \oplus_p Y$ for these spaces.

We note that for all of these norms on $X \times Y$ we obtain that $X \times Y$ is again a Banach space if both X and Y are Banach spaces. If X and Y are inner product spaces then one uses in general the norm (4) as for this choice of norm also the product $X \times Y$ will again be a inner product space with inner product $((x,y),(x',y')) = (x,x')_X + (y,y')_Y$, while none of the norms with $p \neq 2$ preserve the structure of an inner product space.

Deep Dive

We can also consider countable direct sums. Given normed spaces $(X_n)_{n=1}^{\infty}$, and $1 \leq p \leq \infty$, one can form the ℓ^p-direct sums. For $p = \infty$, let

$$X_\infty = \{(x_n) : x_n \in X_n,\ \sup \|x_n\| < \infty\}$$

with the norm $\|(x_n)\|_\infty = \sup \|x_n\|$. For $1 \leq p < \infty$ let

$$X_p = \{(x_n) : x_n \in X_n : \sum_{n=1}^{\infty} \|x_n\|^p < \infty\}$$

with the norm $\|(x_n)\|_p = (\sum_{n=1}^{\infty} \|x_n\|^p)^{1/p}$.

These spaces might be written as $(\oplus_{n=1}^{\infty} X_n)_p$. We could also consider a c_0-sum.

It's a good exercise in seeing if you understand the proofs that ℓ^p forms a Banach space to check these are norms, and that if each X_n is complete so too our the spaces X_p. This time of course all these norms will in general give rise to pairwise non-isomorphic spaces (as can be seen by taking each $X_n = \mathbb{F}$ when you get back the classical sequence spaces ℓ^p). Now you can start asking what sort of spaces you get if you take an infinite ℓ^p product say of a sequence of L^{q_n} spaces!

Sums of subspaces If $X_1, X_2 \subset X$ are subspaces of a normed space X then also

$$X_1 + X_2 := \{x_1 + x_2 : x_1 \in X_1, x_2 \in X_2\}$$

is again a subspace of X, but beware. Just because X_1 and X_2 are closed in X, it does not necessarily follow that $X_1 + X_2$ is closed; see example sheet 1 (question C1) for an example.

Quotients We saw the process of taking quotients to pass from the semi-norm on $\mathscr{L}^p$ to the normed space L^p above. This works generally. Given a vector space X and a semi-norm $|\cdot|$ on X, i.e. a function $|\cdot| : X \to [0,\infty)$ satisfying (N2) and (N3), we can consider the quotient space X/X_0 where $X_0 := \{x \in X : |x| = 0\}$. Then one can define a norm on X/X_0 by defining $\|x + X_0\| := |x|$, see problem sheet 1 for details.

Deep Dive

There are many reasons to be interested in quotient spaces more generally. Suppose X is a normed space, and Y is a subspace of X, when can we put a norm on X/Y? The solution is to define

$$\|x+Y\| = \inf\{\|x+y\| : y \in Y\} = \inf\{\|x-y\| : y \in Y\} = d(x,Y),$$

In general this is only a seminorm as if $\|x+Y\| = 0$, then there is a sequence $y_n \in Y$ with $\|x+y_n\| \to 0$. Noting that $-y_n \in Y$, it follows that $\|x+Y\| = 0$ if and only if x is in the closure $\overline{Y}$ of Y. In this way we get a norm on X/Y precisely when Y is closed.[a].

This will be explored further in C4.1, but it is nice to know that the quotient of a Banach space by a closed subspace is again a Banach space (this is normally proved by showing absolute convergence implies convergence). Since the kernel of a continuous linear map is always closed, one can go on to work out what the right first isomorphism theorem should be in the setting of normed spaces (spoiler alert, there is a subtlety: it will not always be the case that the range space is isomorphic to the domain modulo the kernel, but it works for continuous linear maps between Banach spaces with closed range).

[a] When Y is not closed, you could follow the construction of taking a further quotient of X/Y by the null space of the seminorm. You can check that this gives the same thing as considering the quotient $X/\overline{Y}$, so in practise we only consider quotients by closed subspaces.

2 Inner product spaces and Hilbert spaces

In this section we turn to the important special case when the norm arises from an inner product, leading to the class of Hilbert spaces – one of the most central objects in mathematics. Just as with finite dimensional inner product spaces (and unlike Banach spaces), Hilbert spaces are completely classified upto isometric isomorphism by their *dimension*: the cardinality of an *orthonormal basis* (the appropriate notion of basis in the setting of Hilbert spaces).

2.1 Definitions and basic properties

Definition 2.1. An inner (scalar) product in a linear vector space X over $\mathbb{R}$ is a real-valued function on $X \times X$, denoted as $\langle x, y\rangle$, having the following properties:

(i) *Bilinearity*. For fixed y, $\langle x, y\rangle$ is a linear function of x, and for fixed x, $\langle x, y\rangle$ is a linear function of y.

(ii) *Symmetry*. $\langle x, y\rangle = \langle y, x\rangle$ for all $x, y \in X$.

(iii) *Positivity*. $\langle x, x\rangle > 0$ for $x \neq 0$.

When X is a vector space over $\mathbb{C}$, $\langle x, y\rangle$ is complex-valued and properties (i) and (ii) are replaced by

(i') *Sesquilinearity*. For fixed y, $\langle x, y\rangle$ is a linear function of x, and for fixed x, $\langle x, y\rangle$ is a skewlinear function of y, i.e.
$$\langle ax, y\rangle = a\langle x, y\rangle \text{ and } \langle x, ay\rangle = \bar{a}\langle x, y\rangle \text{ for all } a \in \mathbb{C}, x, y \in X.$$

(ii') *Skew symmetry*. $\langle x, y\rangle = \overline{\langle y, x\rangle}$ for all $x, y \in X$.

Warning. In some textbooks and courses, the sesquilinearity property is reversed: $\langle x, y\rangle$ is required instead to be skewlinear in x and linear in y. This particularly the case when one is coming from a quantum theory viewpoint, when the braket notion $\langle x|y\rangle$ is often used for the inner product.

An inner product $\langle \cdot, \cdot\rangle$ generates a norm, denoted by $\|\cdot\|$, as follows:
$$\|x\| = \langle x, x\rangle^{1/2}.$$

Then the positivity of the norm $\|\cdot\|$ follows from the positivity property (iii), and the homogeneity of $\|\cdot\|$ follows from the bi/sequi-linearity property (i)/(i'). The triangle inequality is a consequence of the Cauchy-Schwartz inequality below. The proof below is the same as in prelims.

Theorem 2.2 (Cauchy-Schwarz inequality)**.** *For $x, y \in X$,*

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Equality holds if and only if x and y are linearly dependent.

Proof. If $y = 0$, the conclusion is clear. Assume henceforth that $y \neq 0$. Replacing x by ax with $|a| = 1$ so that $a\langle x, y\rangle$ is real, we may assume without loss of generality that $\langle x, y\rangle$ is real.

For $t \in \mathbb{R}$, we compute using sesquilinearity and skew symmetry:

$$\|x+ty\|^2 = \langle x+ty, x+ty\rangle = \|x\|^2 + 2t\,\mathrm{Re}\,\langle x, y\rangle + t^2\|y\|^2. \tag{5}$$

By positivity, this quadratic polynomial in t is non-negative for all t. This implies that

$$(\mathrm{Re}\,\langle x, y\rangle)^2 - \|x\|^2\|y\|^2 \leq 0,$$

which gives the desired inequality. If equality holds, then there is some t_0 such that $x + t_0 y = 0$. The conclusion follows. □

Note that the Cauchy-Schwartz identity ensures that the inner product $\langle \cdot, \cdot \rangle$ gives a continuous map $X \times X \to \mathbb{F}$.

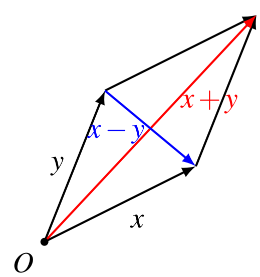
Proposition 2.3. *Let X be an inner product space. Then the* parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \textit{ for all } x, y \in X, \tag{6}$$

holds.

Proof. Set $t = \pm 1$ in (5) and add the resulting identities. □

This is illustrated[15] in $\mathbb{R}^2$ by



In fact the parallelogram law determines whether a norm comes from an inner product. See Sheet 2.A.3 for a proof.[16]

[15] Notice that the identity only involves the vectors x, y and so is verified in the 2-dimensional subspace $\mathrm{Span}(x, y)$ which we know is isometrically isomorphic to $\mathbb{R}^2$ with the usual inner product by means of the Gramm-Schmidt process. So if you've known the parallelogram law as a fact about parallelograms, then you've actually known the real case of the parallelogram law for inner product spaces!

[16] It's not hard to check that the expressions for the inner product in terms of the polarisation identities are the only things that can work: if you know the norm comes from an inner product simply multiply out the right hand sides . The difficulty is seeing that these identities do define an inner product. Note that the polarisation identity can be used in various other situations. For example, it shows that any isometric bijection between inner product spaces necessarily preserves the inner product.

Proposition 2.4. *Let $(X, \|\cdot\|)$ be a normed space satisfying the parallelogram law* (6)*. Then the norm is induced from an inner product, which is given in terms of the norm by means of the* polarisation identity*:*

$$\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \text{ when } \mathbb{F}=\mathbb{R},$$

and

$$\langle x,y\rangle = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2) + \frac{1}{4}i(\|x+iy\|^2 - \|x-iy\|^2) \text{ when } \mathbb{F}=\mathbb{C}.$$

Definition 2.5. A linear vector space with an inner product is called an *inner product space*. If it is complete with the induced norm, it is called a *Hilbert space*.

Given an inner product space, one can complete it with respect to the induced norm.[17] Since the inner product is a continuous function on its factors, it can be extended to the completed space. The completed space is therefore a Hilbert space.

2.2 Examples

Example 2.6. 1. The space $\mathbb{C}^n$ or $\mathbb{R}^n$ is a Hilbert space with the standard inner product

$$\langle x,y\rangle = \sum_{k=1}^{n} x_k \bar{y}_k.$$

2. The space $\ell^2 = \{(x_1, x_2, \ldots) = (x_n) : \sum_{n=1}^{\infty} |x_n|^2 < \infty\}$ is a Hilbert space with the inner product

$$\langle x,y\rangle = \sum_{n=1}^{\infty} x_k \bar{y}_k.$$

3. The space $C[0,1]$ of continuous functions on the interval $[0,1]$ is an incomplete inner product space with the inner product

$$\langle f,g\rangle = \int_0^1 f\,\bar{g}\,dx.$$

You can see this as $C[0,1]$ is dense in $L^2([0,1])$, so can not be complete.

4. Let (Ω,μ) be a measure space, e.g. Ω is a subset of $\mathbb{R}^n$ and μ is the Lebesgue measure. The space $L^2(\Omega,\mu)$ of all complex-valued square integrable functions is a Hilbert space with the inner product

$$\langle f,g\rangle = \int_E f\,\bar{g}\,d\mu.$$

The completeness of $L^2(E,\mu)$ is a special case of the Riesz-Fischer theorem on the completeness of the Lebesgue space $L^p(E,\mu)$.

5. A closed subspace of a Hilbert space is a Hilbert space.

6. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. The space $A^2(\mathbb{D})$ consists of all functions which are square integrable and holomorphic in $\mathbb{D}$ is a closed subspace of $L^2(\mathbb{D})$ and is thus a Hilbert space (known as Bergman space). You are asked to prove this on example sheet 2.[18]

[17] Right now the way we would complete a normed space is as per metric spaces: form the completion of the metric space and then extend both the addition, scalar multiplication and the norm by continuity to give the completion the structure of a Banach space. Fortunately there is a better way, which we might discuss at the end of the course.

[18] The Bergman space is an example of a *reproducing kernel Hilbert space*. Unlike the L^2 spaces, whose elements are equivalence classes of functions on a space, the elements of $A^2(\mathbb{D})$ are functions on $\mathbb{D}$ – elements are equal if and only if they agree exactly. Moreover you can recover the value of $f(z)$ from taking a suitable inner product; see Sheet 2.

7. The space $H^2(\mathbb{T})$ of all functions $f \in L^2(-\pi,\pi)$ whose Fourier series are of the form $\sum_{n\geq 0} a_n e^{inx}$ is a closed subspace of $L^2(-\pi,\pi)$ and is thus a Hilbert space. You will be able to see this by noting that the n-th Fourier coefficient of f is given by $\langle f, e_n\rangle$, where $e_n(x) = \frac{1}{2\pi i}e^{inx}$. In this way $H^2(\mathbb{T})$ is a countable intersection of closed sets, so closed. This space is known as Hardy space, and appears in applications to harmonic analysis.

Examples 2, 6 and 7, and 4 (provided $(\Omega,\mathscr{F},\mu)$ is small enough for $L^2(\Omega)$ to be separable – see a footnote to a deep dive in the previous section – and big enough so that $L^2(\Omega)$ is not finite dimensional) are all isometrically isomorphic. Indeed, as we will see in the next subsection there is a unique infinite dimensional separable Hilbert space. But nevertheless, the different presentations of these Hilbert spaces

2.3 Orthogonality

Definition 2.7. Two vectors x and y in an inner product space X are said to be orthogonal if $\langle x,y\rangle = 0$. For $Y \subset X$, define $Y^\perp$ as the space of all vectors $v \in X$ which are orthogonal to Y, i.e. $\langle v,y\rangle = 0$ for all $y \in Y$. When Y is a subspace of X, $Y^\perp$ is called the *orthogonal complement* of Y in X.

We shall see that a Hilbert space always decomposes as the direct sum of a closed subspace and its orthogonal complement, just as you are familiar with in finite dimensions. First we collect the properties of orthogonal complements that don't require completeness.

Proposition 2.8. *Let Y be a subset of an inner product space X. Then*

(i) *$Y^\perp$ is a closed subspace of X.*

(ii) *$Y \subset Y^{\perp\perp}$.*

(iii) *If $Y \subset Z \subset X$, then $Z^\perp \subset Y^\perp$.*

(iv) *$(\overline{\mathrm{span}\,Y})^\perp = Y^\perp$.*

(v) *If Y and Z are subspaces of X such that $X = Y + Z$ and $Z \subset Y^\perp$, then $Y^\perp = Z$.*

Proof. Most of this is left as an exercise / to be recalled from linear algebra. In (i), to see Y is closed suppose $x_n \in Y^\perp$ has $x_n \to x \in X$. Then for $y \in Y$, we have

$$0 = \langle x_n, y\rangle \to \langle x,y\rangle,$$

so $\langle x,y\rangle = 0$, and hence $x \in Y^\perp$. For (v), take $x \in Y^\perp$ and by hypothesis write $x = y + z$ with $y \in Y$ and $z \in Z$. Then, as $x \in Y^\perp$,

$$0 = \langle x,y\rangle = \langle y,y\rangle + \langle z,y\rangle = \|y\|^2,$$

since $z \in Y^\perp$. Therefore $y = 0$ and $x = z \in Z$, i.e. $Y^\perp \subseteq Z$. □

Our main goal in this section is the following theorem, which we will prove at a bit later:

Theorem 2.9 (Projection theorem)**.** *If Y is a closed subspace of a Hilbert space $\mathscr{H}$, then Y and $Y^\perp$ are complementary subspaces: $\mathscr{H} = Y \oplus Y^\perp$, i.e. every $x \in \mathscr{H}$ can be decomposed uniquely as a sum of a vector in Y and in $Y^\perp$.*

In an inner product space context, we will reserve the $\oplus$ symbol for this orthogonal complementation, i.e. write $X = Y \oplus Z$ when Y, Z are subspaces with $Z = Y^\perp$ and $X + Y = X$.[19].

[19] This is compatible with our use of $\oplus$ for products in the previous section. If we take inner product spaces X, Y and equip $X \times Y$ with the inner product $\langle (x_1,y_1),(x_2,y_2)\rangle = \langle x_1,x_2\rangle + \langle x_2,y_2\rangle$ and identify X with the subspace $\{(x,0) : x \in X\}$ and Y with $\{(0,y) : y \in Y\}$ of $X \times Y$, then $X \times Y$ can be written as $X \oplus Y$.

Deep Dive

More generally, as per linear algebra, we subspaces Y,Z of a vector space are *complemented* if $Y\cap Z=\{0\}$ and $Y+Z=X$. Using the axiom of choice, every subspace of a vector space has a complementary subspace: Take a basis A_Y for Y,[a] and extend it to a basis A_X or X.[b] Then take $Z=\mathrm{Span}(A_X\setminus A_Y)$. Note how this proof is the same as the finite dimensional proof of taking a basis for Y and extending it to a basis for X. However, in a normed space context, we don't learn anything about Z. For example, if X is Banach, and Y is closed, when can one take a complementary subspace Z to be closed (so also a Banach space)?

For this reason in Banach space we say subspaces Y,Z are complemented when they are *closed subspaces* with $Y+Z=X$ and $Y\cap Z=\{0\}$. (In C4.1 this is called topologically complemented, to compare with the notion of algebraic complementation of the previous paragraph).

In a Hilbert space, the projection theorem shows that all closed subspaces are complemented. Strikingly this characterises Hilbert space.

Theorem (Lindenstrass and Tzafriri, 1971)**.** *Let X be a Banach space such that every closed subspace has a closed complement. Then there exists an equivalent norm on X under which it is a Hilbert space.*

[a] using Zorn's lemma to obtain a maximal linearly independent set, which is a basis; see B1.2

[b] using Zorn's lemma again to obtain a maximal linearly independent set containing B_Y.

Before proving the projection theorem, let us collect some consequences.

Corollary 2.10. *If Y is a closed subspace of a Hilbert space $\mathscr{H}$, then $Y=Y^{\perp\perp}$ (which is short hand for $(Y^\perp)^\perp$).*

Proof. We have $\mathscr{H}=Y\oplus Y^\perp=Y^\perp\oplus Y^{\perp\perp}$ from the projection theorem. So $Y\subseteq Y^{\perp\perp}$ with $\mathscr{H}=Y^\perp+Y$. The result follows from Proposition 2.8(v). $\square$

Definition 2.11. The *closed linear* span of a set S in a normed space X is the smallest closed linear subspace of X containing S, i.e. the intersection of all such subspaces. We write $\overline{\mathrm{Span}}(S)$ for this subspace, which is the closure of the span of S.[20]

Proposition 2.12. *Let S be a set in a Hilbert space $\mathscr{H}$. Then $\overline{\mathrm{Span}}(S)=S^{\perp\perp}$.*

Proof. Exercise. $\square$

To prove the projection theorem, we use the following geometrical result.

Theorem 2.13 (Closest point in a closed convex subset)**.** *Let K be a non-empty closed convex*[21] *subset of a Hilbert space $\mathscr{H}$. Then, for every $x\in X$, there is a unique point $k\in K$ which is closer to x than any other points of K, i.e. a unique $k\in K$ with*

$$\|x-k\|=\inf_{y\in K}\|x-y\|.$$

By translating (replace x by 0 and K by $\{k-x:k\in K\}$, the closest point theorem is equivalent to the statement that every non-empty closed convex subset of a Hilbert space has a unique element of minimal norm.

Proof. Let

$$d=\inf_{z\in K}\|x-z\|\geq 0$$

[20] Check that the closure of a subspace is a subspace, so that the closure of the span of S is a closed subspace containing S. Since any subspace containing S contains the span of S, any closed subspace containing S must contain the closure of the span of S.

[21] i.e. if $x,y\in K$ and $0<\lambda<1$, then $\lambda x+(1-\lambda y)\in K$

and $y_n \in K$ be a minimizing sequence, i.e.

$$\lim_{n\to\infty} d_n = d, \qquad d_n = \|x - y_n\|.$$

Applying the parallelogram law (6) to $\frac{1}{2}(x-y_n)$ and $\frac{1}{2}(x-y_m)$ yields

$$\left\|x - \frac{1}{2}(y_n + y_m)\right\|^2 + \frac{1}{4}\|y_n - y_m\|^2 = \frac{1}{2}(d_n^2 + d_m^2).$$

Since K is convex, $\frac{1}{2}(y_n + y_m) \in K$ and so $\left\|x - \frac{1}{2}(y_n + y_m)\right\| \geq d$. This and the above implies that (y_n) is a Cauchy sequence. Let y be the limit of this sequence, which belongs to K as K is closed. We then have by the continuity of the norm that $\|x - y\| = \lim \|x - y_n\| = d$, i.e. y minimises the distance from x.

That y is the unique minimiser follows from the same reasoning above. If y' is also a minimiser, we apply the parallelogram law to $\frac{1}{2}(x-y)$ and $\frac{1}{2}(x-y')$ to obtain

$$d^2 + \frac{1}{4}\|y - y'\|^2 \leq \left\|x - \frac{1}{2}(y + y')\right\|^2 + \frac{1}{4}\|y - y'\|^2 = \frac{1}{2}(\|x - y\|^2 + \|x - y'\|^2) = d^2.$$

This implies that $y = y'$. □

Deep Dive

The closest point theorem also holds for some, but not all, other Banach spaces. As you can have a go at on Sheet 2 (C.2), if the unit ball of a Banach space is *uniformly convex* (think of as 'round enough') then the closest point theorem holds. In particular it is valid for ℓ^p and L^p for $1 < p < \infty$. But the uniqueness portion of the closest point theorem fails for ℓ^1 and ℓ^∞ even in two dimensions.[a] We will see an example of a closed convex subset of a Banach (in fact an affine subspace, i.e. a translation of a subspace) which does not have an element of minimal norm on a problem sheet.

[a] Existence of a closest point in to a closed convex set in finite dimensions is a consequence of compactness; see Section **??**.

Proof of the Projection Theorem. Certainly $Y \cap Y^\perp = \{0\}$. It remains to show that $X = Y + Y^\perp$.

Take any $x \in X$ and, since Y is a non-empty closed convex subset of X, there is a point $y_0 \in Y$ which is closer to x than any other points of Y by Theorem 2.13. To conclude, we show that $x - y_0 \in Y^\perp$.[22] Indeed, for all $y \in Y$ and $t \in \mathbb{R}$, we have

$$\|x - y_0\|^2 \leq \|x - \underbrace{(y_0 - ty)}_{\in Y}\|^2 = \|x - y_0\|^2 + 2t\,\mathrm{Re}\,\langle x - y_0, y\rangle + t^2\,\|y\|^2.$$

It follows that $2t\,\mathrm{Re}\,\langle x - y_0, y\rangle + t^2\,\|y\|^2 \geq 0$ for all $t \in \mathbb{R}$. This implies $\mathrm{Re}\,\langle x - y_0, y\rangle = 0$. This concludes the proof if the scalar field is real.

If the scalar field is complex, we proceed as before with t replaced by it to show that $\mathrm{Im}\,\langle x - y_0, y\rangle = 0$. □

[22] We can see this geometrically for $\mathbb{F} = \mathbb{R}$. Fix some $y \in Y$, and consider the plane $\mathrm{Span}(y, y_0)$. The closest point of x to $\mathrm{Span}(y_0, y)$ is still y_0; but we know from 3-dimensional geometry that the closest point of x to this plane is giving by dropping the perpendicular of x to the plane: hence $x - y_0$ is orthogonal to y.

2.4 Orthonormal bases

Definition 2.14. A subset S of a Hilbert space X is called an orthonormal set if $\|x\| = 1$ for all $x \in S$ and $\langle x, y\rangle = 0$ for all $x \neq y \in S$.

S is called an orthonormal basis (or a complete orthonormal set) for X if S is an orthonormal set and its closed linear span is X.

Theorem 2.15. *Every Hilbert space contains an orthonormal basis.*

Proof. The proof is only examinable when the Hilbert space $\mathscr{H}$ is *separable*, i.e. contains a countable dense subset S. In this case label the elements of S as $y_1, y_2, \ldots$. Applying the Gram-Schmidt process[23] we obtain an orthonormal set $B = \{e_1, e_2, \ldots\}$ (which might terminate after some finite stage) such that, for every n, the span of $\{e_1, \ldots, e_n\}$ contains $y_1, \ldots, y_n$. As $\bar{S} = X$, this implies that $X = \overline{\operatorname{span} B}$, and so X is the closed linear span of B.

Deep Dive

In general we need the axiom of choice – in fact the statement that every Hilbert space has an orthonormal basis is equivalent to the axiom of choice – in the equivalent form of *Zorn's Lemma*. Zorn's Lemma will be described in B1.2 (Set Theory), and shown to be equivalent to the axiom of choice there. It allows us to produce sets which are maximal with respect to certain properties.[a] Let S be a maximal orthonormal set[b] in $\mathscr{H}$. If $\overline{\operatorname{Span}}(S) \neq \mathscr{H}$, then this is a proper closed subset of $\mathscr{H}$, so by the projection theorem, there exists $x \in \mathscr{H}$ orthogonal to $\overline{\operatorname{Span}}(S)$, which we can normalise to have $\|x\| = 1$. Then $S \cup \{x\}$ is orthonormal, contradicting maximality of S. Hence $\overline{\operatorname{Span}}(S) = \mathscr{H}$ and S is an orthonormal basis for $\mathscr{H}$.

[a] Precisely: Given a non-empty partially ordered set $\mathscr{P}$ with the property that every chain $\mathscr{C}$ (i.e. a collection $\mathscr{C} \subset \mathscr{P}$ with the property that for all $x, y \in \mathscr{C}$ either $x \leq y$ or $y \leq x$) has an upper bound (i.e. there exists $z \in \mathscr{P}$ with $x \leq z$ for all $x \in \mathscr{C}$. Then Zorn's Lemma ensures that $\mathscr{P}$ has a maximal element, i.e. some $z \in \mathscr{P}$ with $z \geq x$ for all $x \in \mathscr{P}$. (In B1.2 this will be set out when $\mathscr{P}$ is a collection of sets ordered by inclusion satisfying this property, rather than using the language or partially ordered sets.)

[b] If you do B1.2 it's a good exercise in using Zorn to show this exists

□

Given a finite orthonoromal set $e_1, \ldots, e_n$ in an inner product space X, we can always decompose $x \in X$ as

$$x = \sum_{r=1}^{n} \langle x, e_r\rangle e_r + (x - \sum_{r=1}^{n} \langle x, e_r\rangle e_r),$$

where the first term lies in $\operatorname{Span}(e_1, \ldots, e_n)$ and the second lies in $\operatorname{Span}(e_1, \ldots, e_n)^\perp$. In this way $X = \operatorname{Span}(e_1, \ldots, e_n) \oplus \operatorname{Span}(e_1, \ldots, e_n)^\perp$.[24] The element $\sum_{r=1}^{n} \langle x, e_r\rangle e_r$ is the unique closest point in $\operatorname{Span}(e_1, \ldots, e_n)$ to x (we don't need completeness of X for this as $\operatorname{Span}(e_1, \ldots, e_n)$ is finite dimensional. The following is a consequence of Pythagoras:

Proposition 2.16 (Pythagorean theorem)**.** *Let X be an inner product space and $S = \{x_1, x_2, \ldots, x_m\}$ be a finite orthonormal set in X. For every $x \in X$, there holds*

$$\|x\|^2 = \sum_{n=1}^{m} |\langle x, x_n\rangle|^2 + \left\| x - \sum_{n=1}^{m} \langle x, x_n\rangle x_n \right\|^2.$$

[23] The Gram-Schmidt process is usually applied to a set of finitely many linearly independent vectors yielding an orthogonal basis of the same cardinality. In our setting, we will lose the latter property as the vectors y_i's are not necessarily linearly independent.

[24] We will later see that more generally, all finite dimensional subspaces of normed spaces have closed complements as a consequence of the Hahn–Banach theorem

Corollary 2.17 (Bessel's inequality)**.** *Let X be a Hilbert space and $S = \{x_1, x_2, \dots\}$ be an orthonormal sequence in X. Then, for every $x \in X$, there holds*

$$\sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \leq \|x\|^2.$$

We can characterise when an orthonormal sequence forms a basis in terms of always having equality in Bessel's inequality (this is known as Parseval's identity). The proof is strictly speaking only examinable in B4.2, but we've done all the work, so let's give it here.

Theorem 2.18 (Characterising bases)**.** *Let $\mathscr{H}$ be a Hilbert space and $S = \{e_1, e_2, \dots\}$ be an orthonormal sequence in $\mathscr{H}$. Then the following are equivalent:*

1. *S is an orthonormal basis for $\mathscr{H}$;*
2. *$\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2$ for all $x \in \mathscr{H}$ (i.e. Parseval's identity holds)*
3. *$x = \sum_{n=1}^{\infty} \langle x, e_n\rangle e_n$ for all $x \in \mathscr{H}$;*
4. *$\langle x, y\rangle = \sum_{n=1}^{\infty} \langle x, e_n\rangle \overline{\langle y, e_n\rangle}$, for all $x, y \in \mathscr{H}$.*

In this case the map $\mathscr{H} \to \ell^2$ given by $x \mapsto (\langle x, e_n\rangle)_{n=1}^{\infty}$ is an isometric isomorphism.

Proof. 1⇒3: Note that $\sum_{r=1}^{n} \langle x, e_r\rangle e_r$ is the closest point in $\mathrm{Span}(e_1, \dots, e_n)$ to x (as $x - \sum_{r=1}^{n} \langle x, e_r\rangle$ is orthogonal to $e_1, \dots, e_n$). Since $x \in \mathrm{Span}(e_1, e_2, \dots)$, it follows that

$$\|x - \sum_{r=1}^{n} \langle x, e_r\rangle e_r = d(x, \mathrm{Span}(e_1, \dots, e_n)) \to 0,$$

proving 3. 3 $\Longrightarrow$ 4 is obtained from computing the inner product $\left\langle \sum_{r=1}^{n} \langle x, e_r\rangle e_r, \sum_{s=1}^{n} \langle y, e_s\rangle e_s \right\rangle$ and using continuity of the inner product. 4 $\Longrightarrow$ 2 folows from the definition of the norm in terms of the inner product. Finally for 2 $\Longrightarrow$ 1, if 2 holds, then $\|x - \sum_{r=1}^{n} \langle x, e_r\rangle e_r\| \to 0$ as $n \to \infty$ (by Proposition 2.16, giving 1.

For the last part, condition 2 ensures we have defined an isometric linear map. For surjectivity, given $(\alpha_n) \in \ell^2$, the series $\sum_{n=1}^{\infty} \alpha_n e_n$ is absolutely convergent so converges to x in $\mathscr{H}$, which is then mapped onto (α_n). □

Deep Dive

More generally you can check that if S is an orthonormal basis for a Hilbert space $\mathscr{H}$, then $\mathscr{H}$ is isometrically isomorphic to

$$\ell^2(S) := \{f : S \to \mathbb{F} : \sum_{s \in S} |f(s)|^2 < \infty\},$$

(which is given the inner product you would expect). Here the sum of positive elements over this (potentially uncountable) set is given by

$$\sum_{s \in S} |f(s)|^2 = \sup\{\sum_{s \in F} |f(s)|^2 : F \subset S \text{ is finite}\}$$

(which is exactly the definition you would get from taking the Lebesgue integral on S with counting measure).

Deep Dive

While in a Hilbert space the characterisation above allows us to define a basis as an orthonormal set which has dense linear span, and we learn that every x can be written (uniquely) as $\sum_{r=1}^{n} \alpha_n e_n$ (with convergence in $\mathscr{H}$) this is a feature of Hilbert space. We have to be more careful with the definition of a basis in a Banach space. For $X = C([0,1])$, the sequence $1, x, x^2, \ldots$ is linearly independent, and has dense linear span (as we will see in Section **??**). But it is not true that every $f \in C([0,1])$ can be written as a convergent series $\sum_{n=1}^{\infty} \alpha_n x^n$ (with convergence in $C([0,1])$, with it's canonical sup-norm). Functions which can be written in this way are infinitely differentiable. Therefore $1, x, x^2, \ldots$ does not form a *Schauder basis* for $C([0,1])$. This space does have a Schauder basis: a sequence $f_1, \ldots$ such that every $f \in C([0,1])$ can be written uniquely as $\sum_{n=1}^{\infty} \alpha_n f_n$ for some unique scalars α_n, and further this basis can be taken to consist of polynomial functions. But the firs thing that might come to mind doesn't work. This will be explored further in C4.1; see also the books by Corothers and by Albiac and Kalton.

3 Bounded linear operators between normed vector spaces

Whenever we introduce a class of mathematical objects it is always important to understand the appropriate maps between these objects. In the setting of vector spaces, we study linear maps. In the setting of metric spaces we look at continuous maps, or perhaps contractive, or even isometric maps. For our normed spaces the right maps to consider are the continuous linear maps (as well as contractive and isometric linear maps).

3.1 Boundedness and continuity

Recall that a map $T : V \to W$ between vector spaces is linear if $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ for all $x, y \in X$ and scalars $\lambda, \mu \in \mathbb{F}$. Continuity of a map $T : X \to Y$ is a local property: T is continuous if and only if it is continuous at x for all $x \in X$. But for a linear map, we can use linearity to translate continuity at one point to continuity at all other points, so we only need to check continuity at 0. This leads to the following important proposition.

Proposition 3.1. *Let $T : X \to Y$ be a linear map betwen normed spaces. The following are continuous:*

(i) *T is Lipschitz continuous,*

(ii) *T is continuous,*

(iii) *T is continuous at* 0,

(iv) *there exists $K > 0$ such that $\|T(x)\| \leq K\|x\|$ for all $x \in X$.*

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ are immediate. Suppose that T is continuous at 0. Then there is some $\delta > 0$ such that

$$\|Tx\| = \|Tx - T0\| \leq 1 \text{ for } \|x\| = \delta.$$

It follows that, for any $x \neq 0$,

$$\|Tx\| = \frac{\|x\|}{\delta} T\left(\frac{\delta x}{\|x\|}\right) \leq \frac{\|x\|}{\delta}.$$

Clearly, this continues to holds for $x = 0$ and we can take $K = \frac{1}{\delta}$ in condition (iv).

Finally assume (iv) holds, so let $K > 0$ have $\|T(x)\| \leq K\|x\|$ for all $x \in X$¿ Now we use linearity, to get

$$\|T(x) - T(y)\| = \|T(x - y)\| \leq K\|x - y\|,$$

for all $x, y \in X$. That is T is Lipschitz continuous (with Lipschitz constant at most K). $\square$

The last condition is often the most useful, both for establishing continuity

Definition 3.2. Let X and Y be normed spaces (aways assumed to be over the same field $\mathbb{F}$). Then we say that $T : X \to Y$ is a *bounded linear operator* if T is linear and there exists $K > 0$ so that

$$\|T(x)\|_Y \leq K\|x\|_X \text{ for all } x \in X. \tag{7}$$

Define the *operator norm* of a bounded linear operator $T : X \to Y$ by

$$\|T\| = \inf\{K > 0 : \|T(x)\| \leq K\|x\| \text{ for all } x \in X\}.$$

Write $\mathscr{B}(X,Y)$ for the collection of all bounded linear operators from X to Y.

Warning. T being a bounded linear operator does not mean that $T(X) \subset Y$ is bounded. Indeed, the only linear operator with a bounded image is the trivial operator that maps each $x \in X$ to $T(x) = 0$.

We will often abbreviate the space $\mathscr{B}(X,X)$ of bounded linear operators from a normed space X to itself by $\mathscr{B}(X)$.[25] An important special case is the 'bounded linear functionals', i.e. bounded linear functions from a normed vector space to the corresponding field $\mathbb{F} = \mathbb{R}$ (respectively $\mathbb{F} = \mathbb{C}$ for complex vector spaces) and this so called dual space $X^* := \mathscr{B}(X, \mathbb{F})$ will be discussed in far more detail in chapters **??** and **??**.

Note that the infiumum in the definition of the operator norm is attained, i.e. for a bounded linear operator $T : X \to Y$, we have[26]

$$\boxed{\|T(x)\| \leq \|T\|\|x\| \text{ for all } x \in X.}$$

The set of continuous linear maps between normed spaces is a vector space (by AOL). We check that the operator norm gives $\mathscr{B}(X,Y)$ the structure of a normed space. Needless to say, we shall later be interested in when this is complete. Spoiler alert: $\mathscr{B}(X,Y)$ is complete if and only if Y is complete.

Proposition 3.3. *Let X and Y be normed spaces. Then $\|\cdot\|$ is a norm on $\mathscr{B}(X,Y)$. Also, for $T \in \mathscr{B}(X,Y)$, we have (except in the case when $X = \{0\}$)*

$$\|T\|_{\mathscr{B}(X,Y)} = \sup_{x \in X, x \neq 0} \frac{\|Tx\|}{\|x\|} = \sup_{x \in X, \|x\|=1} \|Tx\| = \sup_{x \in X, \|x\| \leq 1} \|Tx\|.$$

Proof. Note that $\|T\| \geq 0$, and if $\|T\| = 0$, then we have $T(x) = 0$ for all x (by positivity of the norm on Y). Hence $T = 0$.

Let $K = \sup\{\|T(x)\| : \|x\| \leq 1,\ x \in X\}$. Then for $x \neq 0$,

$$\|T(x)\| = \left\|T\left(\frac{x}{\|x\|}\right)\right\| \|x\| \leq K\|x\|,$$

and so $\|T\| \leq K$. But for $x \in X$ with $\|x\| \leq 1$, we have $\|T(x)\| \leq \|T\|$. Taking the supremum over all such x we get $K \leq \|T\|$.

Using this characterisation of the norm, we get

$$\|(\lambda T)\| = \sup\{\|(\lambda T)(x)\| : \|x\| \leq 1\} = |\lambda| \sup\{\|T(x)\| : \|x\| \leq 1\} = |\lambda|\|T\|,$$

[25] In some texts, $\mathscr{B}(X,Y)$ is also denoted as $\mathscr{L}(X,Y)$.

[26] Take a sequence (K_n) satisfying (7) with $K_n \to \|T\|$ and use limits preserve weak inequalities.

for $T \in \mathscr{B}(X,Y)$ and $\lambda \in \mathbb{F}$.

Finally for $S,T \in \mathscr{B}(X,Y)$, and $x \in X$, the triangle inequality (in Y) gives

$$\|(S+T)(x)\| \leq \|S(x)\| + \|T(x)\| \leq (\|S\| + \|T\|)\|x\|,$$

so that $\|S+T\| \leq \|S\| + \|T\|$. □

Remark. Note that if Y is an inner product space, then we have

$$\|T\|_{\mathscr{B}(X,Y)} = \sup\{|\langle Tx, y\rangle| : x \in X, y \in Y, \|x\|_X = \|y\|_Y = 1\}.$$

This is a consequence of (i) and the fact that $\|Tx\|_Y = \sup_{y \in Y, \|y\|_Y = 1} |\langle Tx, y\rangle|$.

Warning. For general bounded linear operators, one cannot expect that there exists $x \in X$ so that $\|Tx\| = M\|x\|$, i.e. the supremum $\sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$ is in general not achieved. Some examples can be found on the problem sheets.

Deep Dive

In the case of bounded linear functionals $f : X \to \mathbb{F}$, whether the supreumum $\|f\| = \sup\{|f(x)| : \|x\| \leq 1\}$ is attained is related to the geometry of the unit ball – a theme you may be getting used to. You'll quickly be able to see using the Riesz representation theorem in Section **??** that this supremum is attained whenever X is a Hilbert space, but actually it works for any uniformly convex Banach space. If you solved Exercise 2.C.2 then you should also be able to prove that if X is a uniformly convex Banach space and $f \in X^*$, then there exists a unique $x \in X$ with $\|x\| = 1$ satisfying $f(x) = \|x\|$. In particular bounded linear functionals on L^p attain their norms for $1 < p < \infty$. (This is a result that we'll also be able to see directly later in the course when we determine the general form of a bounded linear functional on L^p).

We note that for any $T \in \mathscr{B}(X,Y)$ both the kernel $\ker(T) := \{x \in X : T(x) = 0\}$ of T and its image $TX =: \{Tx : x \in X\}$ are subspaces (of X respectively Y), but that while $\ker(T)$ is always closed, as it can be viewed as the preimage of the closed set $\{0\}$ under a continuous operator, the image TX is in general not closed.

3.2 Examples

In order to prove that a map $T : X \to Y$ is a bounded linear operator we need to:

(1) Potentially check that T does map into Y, i.e. $Tx \in Y$ for all $x \in X$;[27]

(2) Verify that T is linear;

(3) Find some M so that for all $x \in X$
$$\|Tx\|_Y \leq M\|x\|_X.$$

Whether (1) is needed will depend on context - is there is a discussion to be had about whether $T(x) \in Y$. In cases where the codomain Y is a space like ℓ^p or L^p, it may be necessary to bound some sum or integral to do this. In that case, most likely the bound you get will directly feed into the proof of (3), and it is worth doing these at the same time. See for example the multiplication by functions on $L^2([0,1])$ example below. In most

[27] A well posed question should be clear whether or not you can assume that the map specified does take values in Y, or whether you are expected to prove this. But in your own work if you write down a map, do make sure you check that it does take values where you say it does!

examples we encounter, linearity will be routine; it will typically be enough just to note briefly why the map is linear[28] but a proof in the spirit of Prelims Linear Algebra does not need to be given unless there is good reason to.

Now let us give some examples.

Shift operators For $1 \leq p \leq \infty$, define the shift operators $L,R : \ell^p \to \ell^p$ by

$$R((x_1,x_2,x_3,\dots)) := (0,x_1,x_2,x_3,\dots) \text{ and } L((x_1,x_2,x_3,\dots)) := (x_2,x_3,x_4,\dots)$$

Here L and R are certainly linear and map ℓ^p into ℓ^p. The map R is isometric, i.e. $\|R(x)\| = \|x\|$ for all $x \in X$, and so certainly bounded with $\|R\| = 1$. The map L is not isometric, as it has kernel $\{(x_n) \in \ell^p : x_1 = 0\}$. But L is bounded as

$$\|L(x)\| = \sum_{n=2}^{\infty} |x_n| \leq \sum_{n=1}^{\infty} |x_n| = \|x\|, \quad x = (x_n) \in \ell^p$$

which shows $\|L\| = 1$. Taking $x = (0,1,0,0,\dots)$, we have $\|x\| = 1$ and $\|L(x)\| = 1$ so $\|L\| = 1$.

As we will see when we look at dual operators in the last section of the course, for $1 \leq p < \infty$ the left shift on ℓ^p is related to the right shift on ℓ^q, where q is the Hölder conjugate of p. You'll be able to exploit this in B4.2, when you compute specta of operators.

You can equally look at shift operators on c_0 with analogous results. Expect a nice relationship between the left and right shift operators on c_0 and the right and left shift operators on ℓ^1.

Co-ordinate projections On ℓ^p, define the co-ordinate projection $\mathrm{ev}_n : \ell^p \to \mathbb{F}$ by $\mathrm{ev}_n(x) = x_n$, i.e. the map which evaluates the sequence in the n-th position. Then ev_n is linear and bounded with $\|\mathrm{ev}_n\| = 1$.[29]

Multiplication by functions on $C[0,1]$ Let $X = C([0,1])$, as always equipped with the supremum norm and let $g \in C^0([0,1])$. Then define $M_g : C([0,1]) \to C([0,1])$ by $M_g(f) = fg$. This does map into $C([0,1])$ as the pointwise product of continuous functions is continuous. Certainly M_g is linear, and for $f \in C([0,1])$, we have

$$\|M_g(f)\|_\infty = \sup_{t\in[0,1]} |f(t)g(t)| \leq \|f\|_\infty \|g\|_\infty,$$

so that M_g is bounded and $\|M_g\| \leq 1$. Taking $f \in C([0,1])$ to be $f(t) = 1$ for all t, we have $\|f\| = 1$ and $M_g(f) = g$ so $\|M_g\| \geq \|g\|_\infty$ and hence $\|M_g\| = \|g\|_\infty$.[30]

Note that there was nothing special about $[0,1]$ here. The same works for $C(K)$ where K is any compact metric space (or compact Hausdorff topological space).

Multiplication by functions on $L^2([0,1]))$ Consider instead $g \in L^\infty([0,1])$ and let $X = L^2([0,1])$ (equipped of course with the L^2 norm). Then we can define a map $M_g : X \to X$ by $M_g(f) = fg$. This time we should note that it is the case that $fg \in L^2([0,1])$ when $f \in L^2([0,1])$. Recalling that for $g \in L^\infty([0,1])$, we have $|g(t)| \leq \|g\|_\infty$ almost everywhere, we get the estimate

$$\int_0^1 |(M_g f)(t)|^2 dt = \int_0^1 |f(t)|^2 |g(t)|^2 dt \leq \|g\|_\infty^2 \int |f(t)|^2 dt$$

[28] For example a sentence like 'T is linear as integration is linear'

[29] As $|x_n| \leq \|x\|_p$, while for the standard element $e_n \in \ell^p$ we have $\|e_n\| = 1$ and $|\mathrm{ev}_n(e_n)| = 1$.

[30] This f is the constant function 1, so I would normally write it as $1 \in C([0,1])$, the function with $1(t) = 1$ for all t (where the 1 on the right hand side lies in $\mathbb{F}$). This notation is useful as $C([0,1])$ is not just a Banach space; it is also an *algebra* with the additional multiplication given by the pointwise multiplication. The constant function 1 is the identity for this multiplication: $1g = g$ for all $g \in C([0,1])$. This is what we used in the calculation above. More on *Banach algebras* in some deep dives.

which both shows $fg \in L^2([0,1])$ and gives the estimate

$$\|M_g f\|_{L^2} \leq \|g\|_{L^\infty} \|f\|_{L^2} \text{ for all } f \in X.$$

By linearity of the integral[31], M_g is linear. Putting all this together, M_g is a bounded linear map from $L^2([0,1])$ to $L^2([0,1])$ and $\|M_g\| \leq \|g\|_{L^\infty}$.

Again we have $\|M_g\| = \|g\|_\infty$. To see this, fix $C < \|g\|_\infty$ (if $\|g\|_\infty = 0$, then $g = 0$ a.e. and hence $M_g(f) = 0$ a.e., and $M_g = 0$). By definition of $\|g\|_\infty$ the set $\Omega_C = \{t \in [0,1] : |g(t)| > C\}$ (which is measurable) has positive measure. Let χ_{Ω_C} denote its indicator function, which lies in $L^2([0,1])$. Then

$$\|M_g \chi_{\Omega_C}\|^2 = \int_{\Omega_C} |g(x)|^2 \geq C^2 \int_{\Omega_C} = C^2 \|\chi_{\Omega_C}\|^2$$

Accordingly $\|M_g\| \geq C$. Since $C < \|g\|_\infty$ was arbitrary $\|M_g\| \geq \|g\|_\infty$.

At the same time one can show that for $g(t) = t$, and any $f \in L^2([0,1])$

$$\|Tf\|_{L^2} < \|f\|_{L^2}$$

(this proof is a nice exercise related to the part A course in integration) so this gives an example of an operator for which the supremum $\sup_{f \neq 0} \frac{\|Tf\|}{\|f\|}$ is not attained for any element of the Banach space $X = L^2([0,1])$.

Deep Dive

Planting seeds for the spectrum of an operator in B4.2, this multiplication operator M_g (for $g(t) = t$) is a, or perhaps the, classic example of a bounded operator on $L^2([0,1])$ with no eigenvalues; yet the spectrum of M_g — those $\lambda \in \mathbb{F}$ for which $M_g - \lambda I$ is not invertible is non-empty. In fact the spectrum in this case is $[0,1]$.

There is a converse to the previous result: If a measurable g is such that $fg \in L^2([0,1])$ for all $f \in L^2([0,1])$, then g is an element of $L^\infty([0,1])$. This is a consequence of the Closed graph theorem, which will be treated in B4.2 Functional Analysis 2. As a consequence of this fact you can show that if $T \in \mathscr{B}(L^2([0,1])$ has $TM_g = M_g T$ for all $g \in L^\infty([0,1])$, then there exists $h \in L^\infty([0,1])$ such that $T = M_h$.

Linear maps between Euclidean Spaces We know that any linear map $T : \mathbb{C}^n \to \mathbb{C}^m$ can be written as

$$Tx = Ax \text{ for some } A \in M_{m \times n}(\mathbb{C}).$$

For the purpose of discussing the operator norm of T, we will equip $\mathbb{C}^n$ with the Euclidean ℓ^2-norm in this section. Certainly from the formula giving matrix multiplication, T is continuous, so bounded.

There are several different norms on the space of matrices, including the analogues of the p-norms on $\mathbb{R}^n$. One that can be useful is the analogue of the Euclidean norm (i.e. of the case $p = 2$) given by

$$\|A\|_2 := \Big(\sum_{i,j} |a_{ij}|^2\Big)^{1/2}$$

which is also called the Frobenius norm or the Hilbert-Schmidt norm and is widely used in Numerical Analysis. A useful property of this norm is that it gives a simple way of obtaining an upper bound on the operator norm of the corresponding map $T : \mathbb{C}^n \to \mathbb{C}^m$ as follows: for $x \in \mathbb{C}^n$ the Cauchy-Schwartz inequality gives

$$\|Tx\|^2 = \sum_{i=1}^m (Ax)_i^2 = \sum_{i=1}^m \Big(\sum_{j=1}^n a_{ij} x_j\Big)^2 \leq \sum_{i=1}^m \Big(\sum_{j=1}^n a_{ij}^2\Big) \cdot \Big(\sum_{j=1}^n x_j^2\Big) = \|A\|_2^2 \|x\|^2.$$

[31] Which takes work in the integration course, but we now just quote

Therefore $\|T\| \leq \|A\|_2$. However, for most matrices we have $\|T\| < \|A\|$. For example the identity operator on $\mathbb{C}^2$ certainly has $\|I\| = 1$, but the Hilbert-Schmidt norm of the associated matrix is 2. In the case when $m = n$, we can make some progress by diagonalisation. If $A = A^*$, i.e. A is hermitian (equal to its own conjugate transpose),[32] then you can diagonalise A, by finding an orthonormal basis of eigenvectors for A (and hence T). It is then straightforward to see that

$$\|T\| = \max\{|\lambda_1|,\dots.|\lambda_n|\}, \quad \lambda_i \text{ the eigenvalues of } A.$$

In general A need not be hermitian, but A^*A always will be. We have that

$$\|T\| = \max\{|\lambda_1|^{1/2},\dots.|\lambda_n|^{1/2}\}, \quad \lambda_i \text{ the eigenvalues of } A^*A.$$

We could do this now – have a go – but we will see it right at the end of the course as a consequence of the *C^*-identity* for bounded operators on a Hilbert space.

Integral operator on $C([0,1])$: Let $X = C([0,1])$ as always be equipped with the sup-norm. Given any $k \in C([0,1] \times [0,1])$ we map each $x \in X$ to the function $Tx : [0,1] \to \mathbb{F}$ that is given by

$$Tx(s) := \int_0^1 k(s,t)x(t)\mathrm{d}t$$

where the integral is well defined as the integrand is bounded (by $\|k\|_\infty \|x\|_\infty$), and $Tx \in C[0,1]$ by, for example, the continuous parameter DCT.[33] The function k is often called an *integral kernel* or a kernel (which is unfortunate as it has nothing to do with the meaning of the word kernel in the context of the kernel of a linear map or homomorphism). Think of Tx as being given by a continuous version of matrix multiplication over the interval.

Then T is linear (as integration is linear) and for any $s \in [0,1]$ we can bound

$$|Tx(s)| \leq \int_0^1 |k(s,t)x(s)|\mathrm{d}s \leq \|k\|_\infty \|x\|_\infty.$$

Therefore T is a bounded linear operator on $C([0,1])$ with $\|T\| \leq \|k\|_\infty$.

An unbounded operator To show that a proposed linear operator $T : X \to Y$ is unbounded, you'll want to find a bounded sequence (x_n) (typically all of norm 1) such that $(T(x_n))$ is unbounded. Here is an example of an unbounded linear functional. Let X be the set of polynomial functions on $[0,1]$ equipped with the sup norm, and let $T : X \to \mathbb{C}$ be given by $T(p) = p'(1)$. Then T is unbounded. Indeed, the polynomial $p_n(t) = t^n$ has $\|p_n\|_\infty = 1$ for all n (as we work over $[0,1]$) but $(Tp_n) = n \to \infty$.

[32] we will have much more to say about the adjoint operation in the last section of the course, both for operators on Hilbert spaces and the dual of an operator between Banach spaces.

[33] Here's the proof. Given $s_0 \in [0,1]$ and any sequence $s_n \to s_0$, we need to show $Tx(s_n) \to Tx(s_0)$. To this end we set $f_n(t) := k(s_n,t)x(t)$ and $f(t) := k(s_0,t)x(t)$ and observe that

- $f_n(t) \to f(t)$ for every $t \in [0,1]$, so in particular $f_n \to f$ a.e.
- $|f_n| \leq g$ on $[0,1]$ for the constant function $g := \|k\|_\infty \|x\|_\infty$ which is of course integrable over the interval $[0,3]$.

Hence, by the dominated convergence theorem of Lebesgue, we have that

$$\lim_{n\to\infty}(Tx)(s_n) = \lim_{n\to\infty}\int_0^1 f_n(t)\mathrm{d}t = \int_0^1 \lim_{n\to\infty} f_n(t)\mathrm{d}t = \int_0^1 f(t)dt = (Tx)(s_0)$$

as claimed. In this case we could get away with a Riemann integral argument as everything in sight is continuous on closed and bounded sets, so you can deduce continuity of Tx usnig uniform continuity of the continuous function k on the compact space $[0,1] \times [0,1]$.

Deep Dive

Perhaps some of you are complaining that the space of polynomials X above is not a Banach space? What is an example of an unbounded linear functional $X \to \mathbb{F}$ when X is Banach? Or indeed an unbounded linear map $X \to Y$ where X is Banach and Y is a normed space?

Firstly, it follows from the axiom of choice that every infinite dimensional normed space X admits an unbounded linear functional. The idea, which will be given in C4.1, is to take an infinite linearly independent set of vectors (x_n) each of norm 1, and extend this arbitrarily to a Hamel basis (using Zorn's lemma). Then you can define a functional $T : X \to \mathbb{F}$ by sending each x_n to n and sending other basis elements to 0. This is a linear map, as linear maps are uniquely determined by their behaviour on a Hamel basis, in just the same way as in prelims linear algebra, and by construction $\|T(x_n)\| \geq n \to \infty$ with $\|x_n\| = 1$ so T is unbounded.

It is possible to find models of ZF without AC for which every linear map from a Banach space to a normed space is bounded.[a] However my take is that the axiom of choice is a true statement when we're studying functional analysis! So what this means is that you won't be writting down any everywhere defined unbounded linear maps on a Banach space any time soonL every linear map $T : X \to Y$ you explicitly construct on a Banach space X is going to be bounded. But beware, that means that you have to define your operator on all elements of the domain, and it must map into a normed space Y, i.e. $T(x) \in Y$ for all $x \in X$. There are many interesting examples of 'densely defined' unbounded operators (and a very interesting theory crucial to formalising quantum mechanics, which we can start to build once we have the closed graph theorem for bounded operators). But

Finally, be in no doubt that you still need to prove your operators are bounded directly. While it's useful to know that without the axiom of choice, it's possible for all everwhere defined linear operators to be bounded, appealing to this deep dive isn't a valid way to proceed in an exam!

[a] Garnir's paper 'Solovay's axiom and Functional Analysis, Springer Lecture Notes in Mathematics, 399, 189-204, 1974' shows that this holds for a model with dependent choice and the hypothesis that every set of reals is Lebesgue measurable.

3.3 Properties of (the space of) bounded linear operators

The space of bounded linear operators is a normed space, so we want to know when it is complete. This happens when the target space is complete.[34] Note how the proof follows the standard 'completeness strategy' discussed in section 1.

Theorem 3.4. *Let X be any normed space and let Y be a Banach space. Then $\mathscr{B}(X,Y)$ (equipped with the operator norm) is complete and therefore is a Banach space.*

Proof. Let (T_n) be a Cauchy-sequence in $\mathscr{B}(X,Y)$. Then for every $x \in X$ we have that

$$\|T_n x - T_m x\| \leq \|T_n - T_m\| \, \|x\| \to 0$$

as $m,n \to \infty$, so $(T_n x)$ is a Cauchy sequence in Y and, as Y is complete, thus converges to some element in Y which we call Tx.

We now show that the resulting map $x \mapsto Tx$ is an element of $\mathscr{B}(X,Y)$ and $T_n \to T$ in $\mathscr{B}(X,Y)$, i.e. $\|T - T_n\| \to 0$.

We first note that the linearity of T_n (and (AOL)) implies that also T is linear. Given any $\varepsilon > 0$ we now let N be so that for $m,n \geq N$ we have $\|T_n - T_m\| \leq \varepsilon$. Given any $x \in X$, continuity of the norm gives

$$\|Tx - T_n x\| = \|\lim_{m\to\infty} T_m x - T_n x\| = \lim_{m\to\infty} \|T_m x - T_n x\| \leq \varepsilon \|x\|.$$

[34] In fact $\mathscr{B}(X,Y)$ is complete if and only if Y is complete; the converse direction will follow from the Hahn-Banach theorem; see sheet 4.

Hence T is bounded (as $\|Tx\| \leq (\|T_n\| + \varepsilon)\|x\|$ for all x) and so an element of $\mathscr{B}(X,Y)$ with $\|T - T_n\| \leq \varepsilon$ for all $n \geq N$, so as $\varepsilon > 0$ was arbitrary we obtain that $T_n \to T$ in the sense of $\mathscr{B}(X,Y)$. $\square$

We note in particular that if X is a Banach-space then the space $\mathscr{B}(X) := \mathscr{B}(X,X)$ of bounded linear operators from X to itself is a Banach space and that for any normed space X the dual space $X^* = \mathscr{B}(X,\mathbb{R})$ (respectively $X^* = \mathscr{B}(X,\mathbb{C})$ if X is a complex vector space) is complete as both $\mathbb{R}$ and $\mathbb{C}$ are complete.

Given any normed spaces X, Y and Z and any linear operators $T \in \mathscr{B}(X,Y)$ and $S \in \mathscr{B}(Y,Z)$ we can consider the composition $ST = S \circ T : X \to Z$ and observe that:

Proposition 3.5. *The composition ST of two bounded linear operators $S \in \mathscr{B}(Y,Z)$ and $T \in \mathscr{B}(X,Y)$ between normed spaces X,Y,Z is again a bounded linear operator and we have*

$$\|ST\|_{\mathscr{B}(X,Z)} \leq \|S\|_{\mathscr{B}(Y,Z)} \|T\|_{\mathscr{B}(X,Y)}.$$

Proof. The only thing we should prove is the estimate,[35] which follows as for $x \in X$, we have

$$\|STx\| = \|S(Tx)\| \leq \|S\|\,\|Tx\| \leq \|S\|\,\|T\|\,\|x\|.$$

$\square$

Remark. The proposition implies in particular that for sequences $T_n \to T$ in $\mathscr{B}(X,Y)$ and $S_n \to S$ in $\mathscr{B}(Y,Z)$ also

$$S_n T_n \to ST \text{ in } \mathscr{B}(X,Z)$$

since

$$\|S_n T_n - ST\| \leq \|(S_n - S)T_n\| + \|S(T_n - T)\| \leq \|S_n - S\|\|T_n\| + \|S\|\|T_n - T\| \to 0$$

where we use in the last step that $\|T_n\|$ is bounded since T_n converges. That is multiplication (i.e. composition) of operators

Deep Dive

Let X be a Banach space. Then the space of bounded linear operators $\mathscr{B}(X)$ is an example of a *unital Banach algebra*. A unital Banach algebra is a Banach space A together with an associative multiplication $A \times A \to A$ which has an identity element 1 with $1x = x1 = x$ for all $x \in A$ (for $\mathscr{B}(X)$, the identity is I_X) such that the multiplication interacts with the Banach space addition and scalar mutliplication in the way you would expect, [a] and satisfying

$$\|ab\| \leq \|a\|\|b\|, \qquad \text{for all } a,b \in A.$$

The last condition, which is Proposition 3.5 for $\mathscr{B}(X)$, shows that the multiplication is jointly continuous. Note that the multiplication need not be commutative the example of composition of operators in $\mathscr{B}(X)$ is not generally commutative.

We have seen some other Banach algebras already: $C(K)$ with pointwise multiplication, and $L^\infty([0,1])$ with pointwise multiplication (defined almost everywhere), are both Banach algebras with the supremum and essential supremum norms. In fact the map $M_\bullet$ sending $g \in C([0,1])$ to the mutliplication operator $M_g \in \mathscr{B}(C([0,1]))$ discussed in the previous section is a Banach algebra *homomorphism*: $M_\bullet$ is linear in g, and preserves the multiplication $M_{gh} = M_g M_h$ (which in this case is $M_h M_g$). In our example we found that $\|M_g\| = \|g\|_\infty$ so $M_\bullet$ is isometric, so certainly bounded. In general just as linear maps need not be automaically bounded, so too Banach algebras homomorphisms are not always bounded (though there are

[35] we are already very familar with the fact that the composition of linear operators is linear, and the composition of continuous maps is continuous

many nice situations where they are)! Here's another unital Banach algebra:

$$\ell^1(\mathbb{Z}) = \{(x_n)_{n\in\mathbb{Z}} : \sum_{n\in\mathbb{Z}} |x_n| < \infty\}$$

with the ℓ^1 norm and convolution multiplication

$$(xy)_n = \sum_r x_r y_{n-r}.$$

What is the identity?

When X is a Banach space, any closed subalgebra[b] $A \subset \mathscr{B}(X)$ containing the identity is a unital Banach algebra. Conversely, if A is a unital Banach algebra with identity 1, consider the homomorphism $M_\bullet : A \to \mathscr{B}(A)$ given by $M_a(b) = ab$. [This generalises the multiplication map on $C([0,1])$.] This is a homomorphism as $(M_aM_b)(c) = M_a(M_b(c)) = a(bc) = (ab)(c) = M_{ab}(c)$ and $\|M_a(b)\| \leq \|a\|\|b\|$ so $M_a \in \mathscr{B}(A)$ with $\|M_a\| \leq \|a\|$ and from taking $b = 1$, we get $\|M_a\| \geq \|a\|/\|1\|$, so M_a is bounded below, and hence the image $\{M_a : a \in \mathscr{B}(A)\}$ is closed in $\mathscr{B}(A)$.

Unital Banach algebras provide the right abstract framework for spectral theory, which we will develop in B4.2 for operators in $\mathscr{B}(X)$. As you do that it's worth going through and seeing that it all works fine in a unital Banach algebra with no real changes to the arguments.

[a] Have a go at axiomising this motivated by the relations you find in $\mathscr{B}(X)$

[b] i.e. a closed subspace also closed under the multiplication

We also note that for operators $T \in \mathscr{B}(X)$ from a normed space X to itself we can consider the composition of T with itself, and more generally powers $T^n = T \circ T \circ \ldots \circ T \in \mathscr{B}(X)$ which, by the above proposition have norm

$$\|T^n\| \leq \|T\|^n.$$

We can use this to define suitable power series of operators.[36]

Example 3.6. Let X be a Banach space and let $A \in \mathscr{B}(X)$. Then[37]

$$\exp(A) := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$$

converges in $\mathscr{B}(X)$ and hence $\exp(A)$ is a well defined element of $\mathscr{B}(X)$.

Proof. We know that

$$\sum_{k=0}^{\infty} \|\frac{1}{k!} A^k\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = \exp(\|A\|) < \infty,$$

i.e. that the series converges absolutely. As X is complete and thus, by Theorem 3.4, also $\mathscr{B}(X)$ is complete we hence obtain from Corollary 1.6 that the series converges. $\square$

[36] The following works equally well for an element of a Banach algebra

[37] Here $A^0 = I$, the identity operator on X.

Deep Dive

This is the starting point of a fundamental tool in studying operators: *functional calculi*. A functional calculus gives a consistent way of defining $f(T)$ for a suitable bounded operators T, and suitable classes of functions $f : D \to \mathbb{C}$, for suitable $D \subset \mathbb{C}$. At the moment you can use Taylor's theorem to extend the example above and define $f(T)$ whenever f is a holomorphic function $\mathbb{C} \to \mathbb{C}$, and also define $f(T)$ when f is given by a power series with radius of convergence exceeding $\|T\|$. The sort of thing you might like is given two functions f and g, to have that $(f \circ g)(T) = f(g(T))$. You'll get a chance to do something like this with power series on exercise sheet 3. This gives you a first functional calculus, but once we've defined the spectrum it's possible to build more sophisticated functional calculus, such as the *holomorphic functional calculus* which allows you to define $f(T)$ whenever f is a holomorphic function on the spectrum of a bounded operator T (or more generally an element in a Banach algebra), or later the continuous and Borel functional calculi, which work for self-adjoint (and more generally normal) operators on a Hilbert space.

3.4 Invertibility

Just as in finite dimensions we shall be interested in when bounded operators are invertible in $\mathscr{B}(X)$, i.e. when a bounded linear operator is bijective and the inverse map is bounded.

Definition 3.7. An element $T \in \mathscr{B}(X)$ is called invertible (short for *invertible in* $\mathscr{B}(X)$) if there exists $S \in \mathscr{B}(X)$ so that $ST = TS = I_X$.[38] When it exists, S is called the inverse of T written T^{-1}.

If we only talk about $T : X \to X$ being 'invertible as a function between sets', we sometimes say that T is algebraically invertible and that a function $S : X \to X$ is an algebraic inverse of T if $ST = TS = I$ (but not necessarily $S \in \mathscr{B}(X)$).

Deep Dive

The Banach isomorphism theorem (a consequence of the Banach open mapping theorem) will tell you that if $T \in \mathscr{B}(X)$ is algebraically invertible and X is a Banach space, then T is invertible. This will be proved in B4.2.

In many applications, including spectral theory which will be discussed in B4.2 Functional Analysis II, the following lemma turns out to be useful to prove that an operator is invertible. The statement should be reminisant of the convergent geometric series from prelims. In fact the proof (when set out in the right way) is also the same telescoping sum argument as prelims.

Lemma 3.8 (Convergence of Neumann-series)**.** *Let X be a Banach space and let $T \in \mathscr{B}(X)$ be so that $\|T\| < 1$. Then the operator $I - T$ is invertible with*

$$(I-T)^{-1} = \sum_{j=0}^{\infty} T^j \in \mathscr{B}(X).$$

Proof of Lemma 3.8. As $\|T\| < 1$ we know that $\sum \|T^k\| \leq \sum \|T\|^k < \infty$ so, by Corollary 1.6, the series converges

$$S_n := \sum_{k=0}^{n} T^k \to S = \sum_{k=0}^{\infty} T^k \text{ in } \mathscr{B}(X).$$

[38] It is necessary that S is a two sided inverse. Going back to our left and right shift operators we have $LR = I$ on ℓ^p but $RL \neq I$.

As

$$(I-T)S_n = I - T + T - T^2 + T^2 - \ldots - T^n + T^n - T^{n+1} = I - T^{n+1}$$

and $\|T^{n+1}\| \leq \|T\|^{n+1} \to 0$, we can pass to the limit $n \to \infty$ in the above expression to obtain that $(I-T)S = I$ and similarly $S(I-T) = I$ so $S = (I-T)^{-1}$. $\square$

Corollary 3.9. *Let X be a Banach space. Then the invertible operators on X are open. Precisely, if $T \in \mathscr{B}(X)$ be invertible, then for any $S \in \mathscr{B}(X)$ with $\|S\| < \|T^{-1}\|^{-1}$ we have that $T - S$ is invertible.*

Proof. Fix invertible $T \in \mathscr{B}(X)$, and let $S \in \mathscr{B}(X)$ have $\|S\| < \|T^{-1}\|^{-1}$. As T is invertible (which by definition means that also $T^{-1} \in \mathscr{B}(X)$) we obtain can write $T - S = T(I - T^{-1}S)$ and note that $T^{-1}S \in \mathscr{B}(X)$ with $\|T^{-1}S\|_{\mathscr{B}(X)} \leq \|T^{-1}\|\|S\| < 1$. By Lemma 3.8 we thus find that $(I - T^{-1}S)$ is invertible with $(I - T^{-1}S)^{-1} = \sum_{j=0}^{\infty}(T^{-1}S)^j \in \mathscr{B}(X)$ and hence $T - S$ is the composition of two invertible operators and thus invertible, compare also A.1 on Problem Sheet 2.

Since $T^{-1} \neq 0$, $\|T^{-1}\| \neq 0$, and it follows that the invertible operators are open. $\square$

Notice that if $T \in \mathscr{B}(X)$ is invertible then for $x \in X$,

$$\|x\| = \|T^{-1}T(x)\| \leq \|T^{-1}\|\|T(x)\|.$$

This suggests:

Definition 3.10. Let X, Y be normed spaces and $T \in \mathscr{B}(X,Y)$. Say that T is *bounded below* if there exists $C > 0$ such that

$$\|T(x)\| \geq C\|x\| \quad \text{for all } x \in X.$$

We have seen that invertible operators are bounded below (by the norm of the inverse). Hence being bounded below is necessary for invertibility, and for an algebraically invertible $T \in \mathscr{B}(X)$, we have that T is invertible if and only if it is bounded below. An operator which is bounded below is certainly injective. When the domain is complete, operators which are bounded below also have closed range.

Proposition 3.11. *Let X be a Banach space, Y be a normed space and $T \in \mathscr{B}(X,Y)$ be bounded below. Then $T(X)$ is closed in Y.*

Proof. Suppose that (x_n) is a sequence with $T(x_n) \to y \in Y$. Let $C > 0$ be such that $\|Tx\| \geq C\|x\|$ for all x, so that $\|x_n - x_m\| \leq C^{-1}\|Tx_n - Tx_m\| \to 0$, as (Tx_n) is Cauchy. By completeness of X, there exists $x \in X$ with $x_n \to x$. By continuity of T, $Tx_n \to Tx$, so $y = Tx \in T(X)$. $\square$