B1.1 Logic

Sheet 2 — MT25

You may freely use the equivalences of Exercise 3.14 on this sheet, except where you are specifically asked to prove them,

Section A

1. Let ϕ be the formula $((p_0 \to p_1) \land (p_1 \to p_2))$. Find a formula in disjunctive normal form which is logically equivalent to ϕ and has precisely three disjuncts.

2. Prove that for any formulas α, β of $\mathcal{L}_0$, the following formulas are theorems of the system L_0. You may use the deduction theorem.

   (a) $(\neg\alpha \to (\alpha \to \beta))$

   (b) $((\neg\alpha \to \alpha) \to (\neg\alpha \to \beta))$

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Section B

3. Prove that the following logical equivalences hold for any $n \geq 1$ and any formulas $\psi, \phi_1, \ldots, \phi_n \in \mathrm{Form}(\mathcal{L}_{\mathrm{prop}})$.

   (a) $\neg \bigwedge_{i=1}^{n} \phi_i \vDash\Dashv \bigvee_{i=1}^{n} \neg\phi_i$.

   (b) $(\psi \vee \bigwedge_{i=1}^{n} \phi_i) \vDash\Dashv \bigwedge_{i=1}^{n}(\psi \vee \phi_i)$.

   (c) $(\psi \wedge \bigvee_{i=1}^{n} \phi_i) \vDash\Dashv \bigvee_{i=1}^{n}(\psi \wedge \phi_i)$.

4. Prove that every formula is logically equivalent to one in conjunctive normal form[1]. (You may use results from lectures, including the corresponding result for disjunctive normal form, Theorem 3.19.)

5. (a) Show that $\mathcal{L}_{\mathrm{prop}}[\vee, \wedge, \to]$ is **not** adequate.

   (b) Find all possible truth tables for a binary connective $\star$ such that $\mathcal{L}_{\mathrm{prop}}[\star]$ is adequate. Justify your answer.

6. Prove that for any formulas α, β of $\mathcal{L}_0$, the following formulas are theorems of the system L_0. You may use the deduction theorem.

   (a) $(\neg\neg\alpha \to \alpha)$

   (b) $(\alpha \to \neg\neg\alpha)$

   (c) $((\alpha \to \beta) \to (\neg\beta \to \neg\alpha))$

   (d) $((\alpha \to \beta) \to ((\neg\alpha \to \beta) \to \beta))$

7. The Four Colour Theorem asserts that on any "reasonable" map showing finitely many countries, each country may be coloured either red, green, blue, or yellow in such a way that no two neighbouring countries get the same colour. Assuming this fact, use the Compactness Theorem of propositional logic to deduce that such a colouring also exists for any reasonable map with countably infinitely many countries. (Assume that deleting countries from a reasonable map yields a reasonable map; this is all you need to know about the definition of "reasonable".[2])

[1] Conjunctive normal form (CNF) is defined analogously to disjunctive normal form: a formula in CNF is a conjunction of disjunctions of propositional variables and negated propositional variables.

[2] For the curious, we can define "reasonable" as follows: it must be possible to draw a point inside each country (the country's *capital*) and plane curves (*roads*) between the capitals of neighbouring countries such that no two roads intersect at a point which is not a capital.

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Section C

8. A *graph* $\langle V; R\rangle$ consists of a set V and a symmetric irreflexive binary relation R on V. (In the terminology of first-order logic, a non-empty graph is precisely a model of $\forall x \forall y ((xRy \leftrightarrow yRx) \wedge \neg xRx)$.)

   (a) Prove the Infinite Ramsey Theorem: any infinite graph $\langle V; R\rangle$ contains an infinite *homogeneous* subset, i.e. an infinite subset $X \subseteq V$ such that either any two distinct elements of X are related by R, or no two distinct elements of X are related by R.

   (b) Apply the compactness theorem of propositional logic to deduce the Finite Ramsey Theorem: for any $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that any graph of size at least n contains a homogeneous subset of size k.

9. The *Yablo paradox* can be stated informally as: "The following sentences are all false. The following sentences are all false. The following. . . ".

   We could formalise this by adding infinitary conjunctions to $\mathcal{L}_{\text{prop}}$, true if every conjunct is true, and considering

   $$\Gamma := \{(p_i \leftrightarrow \bigwedge_{j>i} \neg p_j) : i \in \mathbb{N}\}.$$

   Show that Γ is not satisfiable but every finite subset is satisfiable, so the compactness theorem fails in this logic.

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