$\mathcal{L}$ is a first-order language.

An $\mathcal{L}$-theory is a set of $\mathcal{L}$-sentences

## Compactness Theorem:

If an $\mathcal{L}$-theory $T$ is finitely satisfiable, then it is satisfiable.

We say that a theory $T$ is *finitely satisfiable* if every finite subset of $T$ is satisfiable. We will show that every finitely satisfiable theory $T$ is satisfiable. To do this, we must build a model of $T$. The main idea of the construction is that we will add enough constants to the language so that every element of our model will be named by a constant symbol. The following definition will give us sufficient conditions to construct a model from the constants.

**Definition** We say that an $\mathcal{L}$-theory $T$ has the *witness property* if whenever $\phi(v)$ is an $\mathcal{L}$-formula with one free variable $v$, then there is a constant symbol $c \in \mathcal{L}$ such that $T \models (\exists v\ \phi(v)) \to \phi(c)$.

An $\mathcal{L}$-theory $T$ is *maximal* if for all $\phi$ either $\phi \in T$ or $\neg\phi \in T$.

Our proof will frequently use the following simple lemma.

**Lemma 1** *Suppose $T$ is a maximal and finitely satisfiable $\mathcal{L}$-theory. If $\Delta \subseteq T$ is finite and $\Delta \models \psi$, then $\psi \in T$.*

**Proof** If $\psi \notin T$, then, because $T$ is maximal, $\neg\psi \in T$. But then $\Delta \cup \{\neg\psi\}$ is a finite unsatisfiable subset of $T$, a contradiction.

**Lemma 2** *Suppose that $T$ is a maximal and finitely satisfiable $\mathcal{L}$-theory with the witness property. Then, $T$ has a model. In fact, if $\kappa$ is a cardinal and $\mathcal{L}$ has at most $\kappa$ constant symbols, then there is $\mathcal{M} \models T$ with $|\mathcal{M}| \leq \kappa$.*

**Proof** Let $\mathcal{C}$ be the set of constant symbols of $\mathcal{L}$. For $c, d \in \mathcal{C}$, we say $c \sim d$ if $T \models c = d$.

**Claim 1** $\sim$ is an equivalence relation.

Clearly, $c = c$ is in $T$. Suppose that $c = d$ and $d = e$ are in $T$. By Lemma 2.1.6, $d = c$ and $c = e$ are in $T$.

The universe of our model will be $M = \mathcal{C}/\sim$, the equivalence classes of $\mathcal{C}$ mod $\sim$. Clearly, $|M| \leq \kappa$. We let $c^*$ denote the equivalence class of $c$ and interpret $c$ as its equivalence class, that is, $c^{\mathcal{M}} = c^*$. Next we show how to interpret the relation and function symbols of $\mathcal{L}$.

Suppose that $R$ is an $n$-ary relation symbol of $\mathcal{L}$.

**Claim 2** Suppose that $c_1, \ldots, c_n, d_1, \ldots, d_n \in \mathcal{C}$, and $c_i \sim d_i$ for $i = 1, \ldots, n$, then, $R(\bar{c}) \in T$ if and only if $R(\bar{d}) \in T$.

Because $c_i = d_i \in T$ for $i = 1, \ldots, n$, by Lemma 1, if one of $R(\bar{c})$ and $R(\bar{d})$ is in $T$, then both are in $T$.

We will interpret $R$ as

$$R^{\mathcal{M}} = \{(c_1^*, \ldots, c_n^*) : R(c_1, \ldots, c_n) \in T\}.$$

By Claim 2, $R^{\mathcal{M}}$ is well-defined.

Suppose that $f$ is an $n$-ary function symbol of $\mathcal{L}$ and $c_1, \ldots, c_n \in \mathcal{C}$. Because $\emptyset \models \exists v\, f(c_1, \ldots, c_n) = v$ and $T$ has the witness property, by Lemma 1, there is $c_{n+1} \in \mathcal{C}$ such that $f(c_1, \ldots, c_n) = c_{n+1} \in T$. As above, if $d_i \sim c_i$ for $i = 1, \ldots, n+1$, then $f(d_1, \ldots, d_n) = d_{n+1} \in T$. Moreover, because $f$ is a function symbol, if $e_i \sim c_i$ for $i = 1, \ldots n$ and $f(e_1, \ldots, e_n) = e_{n+1} \in T$, then $e_{n+1} \sim c_{n+1}$. Thus, we get a well-defined function $f^{\mathcal{M}} : M^n \to M$ by

$$f^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^* \text{ if and only if } f(c_1, \ldots, c_n) = d \in T.$$

This completes the description of the structure $\mathcal{M}$. Before showing that $\mathcal{M} \models T$, we must show that terms behave correctly.

**Claim 3** Suppose that $t$ is a term using free variables from $v_1, \ldots, v_n$. If $c_1, \ldots, c_n, d \in \mathcal{C}$, then $t(c_1, \ldots, c_n) = d \in T$ if and only if $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$.

($\Rightarrow$) We first prove, by induction on terms, that if $t(c_1, \ldots, c_n) = d \in T$, then $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$. If $t$ is a constant symbol $c$, then $c = d \in T$ and $c^{\mathcal{M}} = c^* = d^*$.

If $t$ is the variable $v_i$, then $c_i = d \in T$ and $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = c_i^* = d^*$.

Suppose that the claim is true for $t_1, \ldots, t_m$ and $t$ is $f(t_1, \ldots, t_m)$. Using the witness property and Lemma 2.1.6, we can find $d, d_1, \ldots, d_n \in \mathcal{C}$ such that $t_i(c_1, \ldots, c_n) = d_i \in T$ for $i \leq m$ and $f(d_1, \ldots, d_m) = d \in T$. By our induction hypothesis, $t_i^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d_i^*$ and $f^{\mathcal{M}}(d_1^*, \ldots, d_m^*) = d^*$. Thus $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$.

($\Leftarrow$) Suppose, on the other hand, than $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = d^*$. By the witness property and Lemma 1, there is $e \in \mathcal{C}$ such that $t(c_1, \ldots, c_n) = e \in T$. Using the ($\Rightarrow$) direction of the proof, $t^{\mathcal{M}}(c_1^*, \ldots, c_n^*) = e^*$. Thus, $e^* = d^*$ and $e = d \in T$. By Lemma 2.1.6, $t(c_1, \ldots, c_n) = d \in T$.

**Claim 4** For all $\mathcal{L}$-formulas $\phi(v_1, \ldots, v_n)$ and $c_1, \ldots, c_n \in \mathcal{C}$, $\mathcal{M} \models \phi(\bar{c}^*)$ if and only if $\phi(\bar{c}) \in T$.

We prove this claim by induction on formulas.

Suppose that $\phi$ is $t_1 = t_2$. By Lemma 1 and the witness property, we can find $d_1$ and $d_2$ such that $t_1(\bar{c}) = d_1$ and $t_2(\bar{c}) = d_2$ are in $T$. By Claim 3, $t_i^{\mathcal{M}}(\bar{c}^*) = d_i^*$ for $i = 1, 2$. Then

$$\begin{aligned}\mathcal{M} \models \phi(\bar{c}^*) &\Leftrightarrow d_1^* = d_2^* \\ &\Leftrightarrow d_1 = d_2 \in T \\ &\Leftrightarrow t_1(\bar{c}) = t_2(\bar{c}) \in T \text{ by Lemma } 1.\end{aligned}$$

Suppose that $\phi$ is $R(t_1, \ldots, t_m)$. Because $T$ has the witness property, by Lemma 1 there are $d_1, \ldots, d_m \in \mathcal{C}$ such that $t_i(\bar{c}) = d_i \in T$ and, Claim 4, $t_i^{\mathcal{M}}(\bar{c}^*) = d_i^*$ for $i = 1, \ldots, m$. Thus,

$$\begin{aligned}\mathcal{M} \models \phi(\bar{c}^*) &\Leftrightarrow \bar{d}^* \in R^{\mathcal{M}} \\ &\Leftrightarrow R(\bar{d}) \in T \\ &\Leftrightarrow \phi(\bar{c}) \in T \text{ by Lemma } 1.\end{aligned}$$

Suppose that the claim is true for $\phi$. If $\mathcal{M} \models \neg\phi(\bar{c}^*)$, then $\mathcal{M} \not\models \phi(\bar{c}^*)$. By the induction hypothesis, $\phi(\bar{c}) \notin T$. Thus by maximality, $\neg\phi(\bar{c}) \in T$. On the other hand, if $\neg\phi(\bar{c}) \in T$, then, because $T$ is finitely satisfiable, $\phi(\bar{c}) \notin T$. Thus, by induction, $\mathcal{M} \not\models \phi(\bar{c}^*)$ and $\mathcal{M} \models \neg\phi(\bar{c}^*)$.

Suppose that the claim is true for $\phi$ and $\psi$. Then

$$\begin{aligned}\mathcal{M} \models (\phi \wedge \psi)(\bar{c}^*) &\Leftrightarrow \phi(\bar{c}) \in T \text{ and } \psi(\bar{c}) \in T \\ &\Leftrightarrow (\phi \wedge \psi)(\bar{c}) \in T \text{ by Lemma } 1.\end{aligned}$$

Suppose that $\phi$ is $\exists v\ \psi(v)$ and the claim is true for $\psi$. If $\mathcal{M} \models \psi(d^*, \bar{c}^*)$, then, by the inductive assumption, $\psi(d, \bar{c}) \in T$ and $\exists v\ \psi(v, \bar{c}) \in T$, by 1. On the other hand if $\exists v\ \psi(v, \bar{c}) \in T$, then by the witness property and Lemma 1, $\psi(d, \bar{c}) \in T$ for some $c$. By induction, $\mathcal{M} \models \psi(d^*, \bar{c}^*)$ and $\mathcal{M} \models \exists v\ \psi(v, \bar{c}^*)$.

This completes the induction. In particular, we have $\mathcal{M} \models T$, as desired.

The following lemmas show that any finitely satisfiable theory can be extended to a maximal finitely satisfiable theory with the witness property.

**Lemma 3** *Let $T$ be a finitely satisfiable $\mathcal{L}$-theory. There is a language $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a finitely satisfiable $\mathcal{L}^*$-theory such that any $\mathcal{L}^*$-theory extending $T^*$ has the witness property. We can choose $\mathcal{L}^*$ such that $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.*

**Proof** We first show that there is a language $\mathcal{L}_1 \supseteq \mathcal{L}$ and a finitely satisfiable $\mathcal{L}_1$-theory $T_1 \supseteq T$ such that for any $\mathcal{L}$-formula $\phi(v)$ there is an $\mathcal{L}_1$-constant symbol $c$ such that $T_1 \models (\exists v\ \phi(v)) \rightarrow \phi(c)$. For each $\mathcal{L}$-formula $\phi(v)$, let $c_\phi$ be a new constant symbol and let $\mathcal{L}_1 = \mathcal{L} \cup \{c_\phi : \phi(v) \text{ an } \mathcal{L}\text{-formula}\}$. For each $\mathcal{L}$-formula $\phi(v)$, let $\Theta_\phi$ be the $\mathcal{L}_1$-sentence $(\exists v\ \phi(v)) \rightarrow \phi(c_\phi)$. Let $T_1 = T \cup \{\Theta_\phi : \phi(v) \text{ an } \mathcal{L}\text{-formula}\}$.

**Claim** $T_1$ is finitely satisfiable.

Suppose that $\Delta$ is a finite subset of $T_1$. Then, $\Delta = \Delta_0 \cup \{\Theta_{\phi_1}, \ldots, \Theta_{\phi_n}\}$, where $\Delta_0$ is a finite subset of $T$. Because $T$ is finitely satisfiable, there is

$\mathcal{M} \models \Delta_0$. We will make $\mathcal{M}$ into an $\mathcal{L} \cup \{c_{\phi_1}, \ldots, c_{\phi_n}\}$-structure $\mathcal{M}'$. Because we will not change the interpretation of the symbols of $\mathcal{L}$, we will have $\mathcal{M}' \models \Delta_0$. To do this, we must show how to interpret the symbols $c_{\phi_i}$ in $\mathcal{M}'$. If $\mathcal{M} \models \exists v\ \phi(v)$, choose $a_i$ some element of $M$ such that $\mathcal{M} \models \phi(a_i)$ and let $c_{\phi_i}^{\mathcal{M}'} = a_i$. Otherwise, let $c_{\phi_i}^{\mathcal{M}'}$ be any element of $\mathcal{M}$. Clearly, $\mathcal{M}' \models \Theta_{\phi_i}$ for $i \leq n$. Thus, $T_1$ is finitely satisfiable.

We now iterate the construction above to build a sequence of languages $\mathcal{L} \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \ldots$ and a sequence of finitely satisfiable $\mathcal{L}_i$-theories $T \subseteq T_1 \subseteq T_2 \subseteq \ldots$ such that if $\phi(v)$ is an $\mathcal{L}_i$-formula, then there is a constant symbol $c \in \mathcal{L}_{i+1}$ such that $T_{i+1} \models (\exists v \phi(v)) \to \phi(c)$.

Let $\mathcal{L}^* = \bigcup \mathcal{L}_i$ and $T^* = \bigcup T_i$. By construction, $T^*$ has the witness property. If $\Delta$ is a finite subset of $T^*$, then $\Delta \subseteq T_i$ for some $i$. Thus, $\Delta$ is satisfiable and $T^*$ is finitely satisfiable.

If $|\mathcal{L}_i|$ is the number of relation, function and constant symbols in $\mathcal{L}_i$, then there are at most $|\mathcal{L}_i| + \aleph_0$ formulas in $\mathcal{L}_i$. Thus, by induction, $|\mathcal{L}^*| = |\mathcal{L}| + \aleph_0$.

**Theorem** *If $T$ is a finitely satisfiable $\mathcal{L}$-theory and $\kappa$ is an infinite cardinal with $\kappa \geq |\mathcal{L}|$, then there is a model of $T$ of cardinality at most $\kappa$.*

**Proof** By Lemma 3, we can find $\mathcal{L}^* \supseteq \mathcal{L}$ and $T^* \supseteq T$ a finitely satisfiable $\mathcal{L}^*$-theory such that any $\mathcal{L}^*$-theory extending $T^*$ has the witness property and the cardinality of $\mathcal{L}^*$ is at most $\kappa$. By Prop 2.5 Notes, we can find a maximal finitely satisfiable $\mathcal{L}^*$-theory $T' \supseteq T^*$. Because $T'$ has the

called also complete

witness property, Lemma 2 ensures that there is $\mathcal{M} \models T$ with $|M| \leq \kappa$.