Professor Arguz **B3.2 Geometry of Surfaces** **MT 2025**

Problem Sheet 1 - parts A and C solutions

Part A.

1. Show that the following topological spaces (with natural subspace topologies induced from $\mathbb{R}$) are homeomorphic.

a) The interval $[0,1]$ and the interval $[3,8]$.

b) The interval $(-1,1)$ and the real line.

Solution.

a) Let $f : [0,1] \to [3,8]$ be the map defined by $f(t) = 3 + 5t$. Then, f is continuous and bijective with continuous inverse $f^{-1}(t) = \frac{t-3}{5}$. Therefore f is a homeomorphism.

b) First, note that $(-1,1)$ is homeomorphic to $(-\frac{\pi}{2}, \frac{\pi}{2})$: a homeomorphism is given by $f(t) = \frac{\pi}{2}t$. Moreover, the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ is homeomorphic to $\mathbb{R}$, via the tangent function $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$.

3. Let $\mathbb{S}^2$ denote the sphere, and let X be a compact, connected surface. Show that the connected sum $S\#X$ is homeomorphic to X.

Solution. Note that

$$\chi\left(\mathbb{S}^2\#X\right) = \chi\left(\mathbb{S}^2\right) + \chi\left(X\right) - 2 = 2 + \chi\left(X\right) - 2 = \chi\left(X\right).$$

So $\mathbb{S}^2\#X$ and X have the same Euler characteristic. And $\mathbb{S}^2\#X$ is orientable if and only if X is orientable. Hence by the classification theorem $\mathbb{S}^2\#X$ and X are homeomorphic.

2. Show that there is a quotient map $q : (-2,2) \to [-1,1]$, but not a quotient map $p : [-2,2] \to (-1,1)$.

Solution. Let $\sim$ be the equivalence relation on $(-2,2)$ given by $x \sim y$ if and only if $x = y$ or $x, y \leq -1$ or $x, y \geq 1$. Then, $(-2,2)/\sim$ is homeomorphic to $[-1,1]$. On the other hand, since the image of a compact set under a continuous map should be compact, there is no continuous surjection, in particular no quotient map, $p : [-2,2] \to (-1,1)$.

Part C.

1. Show that the closed disc of radius 1 in $\mathbb{R}^2$ and the closed square in $\mathbb{R}^2$ are homeomorphic.

Solution. We will write a homeomorphism between the closed unit disc

$$B = \{x \in \mathbb{R}^2 : |x - (0,0)| \leq 1\}$$

and the square $Q = [-1,1] \times [-1,1]$. Consider polar coordinates $(r,\theta) \in [0,\infty) \times [0,2\pi)$ on $\mathbb{R}^2$. Note that, for each value of $\theta \in [0,2\pi)$, there is exactly one value of $r_B(\theta)$ such that $(r_B(\theta),\theta) \in \partial B$ (i.e. $r_B(\theta) = 1$), and exactly one value $r_Q(\theta)$ such that $(r_Q(\theta),\theta) \in \partial Q$. Let $g : [0,2\pi) \to (0,\infty)$ be the function defined as $g(\theta) = \frac{r_Q(\theta)}{r_B(\theta)} = r_Q(\theta)$, which is continuous. Then, define Then, let $f : B \to Q$ to be the function defined as $f(r,\theta) = (g(\theta)r,\theta)$. The function f is a bijection from B to Q with inverse $f^{-1}(r,\theta) = (\frac{r}{g(\theta)})$. Note that f and f^{-1} are continuous since g is continuous, hence they are homeomorphisms.