B1.1 Logic
Sheet 3 — MT25

Section A

1. State and prove the unique readability theorem for predicate calculus
   (a) for terms,
   (b) for atomic formulas,
   (c) and for all formulas.

2. Let $\mathcal{L} := \{+, \cdot, <, 0, 1\}$ where $+, \cdot$ are binary function symbols, $<$ is a binary relation symbol, and $0, 1$ are constant symbols.

   Consider the ordered field of real numbers $\mathcal{R}$ as an $\mathcal{L}$-structure.

   Let $\mathcal{L}' := \mathcal{L} \cup \{h\}$ where h is a unary function symbol, and let $\mathcal{R}'$ be the $\mathcal{L}'$-structure obtained by expanding $\mathcal{R}$ by an interpretation $h^{\mathcal{R}'} : \mathbb{R} \to \mathbb{R}$ of h. Find $\mathcal{L}'$-formulas ϕ and ψ such that:
   (a) $\mathcal{R}' \vDash \phi$ iff $h^{\mathcal{R}'}$ is continuous.
   (b) $\mathcal{R}' \vDash \psi$ iff $h^{\mathcal{R}'}$ is differentiable.

3. Let a be an assignment in an $\mathcal{L}$-structure $\mathcal{M}$, let t be an $\mathcal{L}$-term, and let $i \in \mathbb{N}$. Show:
   (a) If u is an $\mathcal{L}$-term, then $a[t/x_i](u) = a(u[t/x_i])$.
   (b) If ϕ is an atomic $\mathcal{L}$-formula, then $\mathcal{M} \vDash_a \phi[t/x_i] \iff \mathcal{M} \vDash_{a[t/x_i]} \phi$.

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Section B

4. Let $\mathcal{L}$ be a language, and let $\alpha, \beta \in \mathrm{Form}(\mathcal{L})$. Assume the variable x_i has no free occurrence in α (i.e. $x_i \notin \mathrm{Free}(\alpha)$).

   In this question, you may cite Lemma 7.15 on the irrelevance of assignment of bound variables, but should otherwise argue directly from the definition of $\mathcal{M} \vDash_a \forall x_j \phi$ and the corresponding characterisation of $\mathcal{M} \vDash_a \exists x_j \phi$.

   (a) Prove
   $$\vDash (\forall x_i(\alpha \to \beta) \leftrightarrow (\alpha \to \forall x_i \beta));$$

   (b) Use part (a) to show
   $$\vDash ((\alpha \vee \forall x_i \beta) \leftrightarrow \forall x_i(\alpha \vee \beta)).$$

   (c) Show that for any γ we have
   $$\vDash ((\exists x_i \gamma \vee \exists x_i \beta) \leftrightarrow \exists x_i(\gamma \vee \beta)),$$
   and deduce that for α as above we have
   $$\vDash ((\alpha \vee \exists x_i \beta) \leftrightarrow \exists x_i(\alpha \vee \beta)).$$

5. Let $\mathcal{L} = \{P\}$ with P a binary relation symbol. By exhibiting three $\mathcal{L}$-structures, show that none of the following sentences is a logical consequence of the other two.

   (i) $\forall x \forall y \forall z(P(x,y) \to (P(y,z) \to P(x,z)))$,

   (ii) $\forall x \forall y(P(x,y) \to (P(y,x) \to x \doteq y))$,

   (iii) $(\forall x \exists y P(x,y) \to \exists y \forall x P(x,y))$.

6. Consider the language $\mathcal{L} = \{f\}$ where f is a unary function symbol.

   (a) Write down sentences ϕ and ψ of $\mathcal{L}$ such that for any $\mathcal{L}$-structure $\mathcal{A} = \langle A; f^{\mathcal{A}} \rangle$,

   (i) $\mathcal{A} \vDash \phi$ if and only if $f^{\mathcal{A}}$ is one-one;

   (ii) $\mathcal{A} \vDash \psi$ if and only if $f^{\mathcal{A}}$ is onto.

   (b) Write down an $\mathcal{L}$-sentence χ which is satisfied by some $\mathcal{L}$-structure with an infinite domain but is false in every $\mathcal{L}$-structure with a finite domain. What can you say about the sizes of the domains of the models of the sentence $\neg\chi$?

   (c) Write down an $\mathcal{L}$-sentence ρ such that a non-empty finite set A is the domain of an $\mathcal{L}$-structure satisfying ρ if and only if $|A|$ is even. What can you say about the sizes of the domains of the models of the sentence $\neg\rho$?

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7. Let $\mathcal{M}$ and $\mathcal{N}$ be isomorphic $\mathcal{L}$-structures, $\mathcal{M} \cong \mathcal{N}$. Let σ be an $\mathcal{L}$-sentence. Show that $\mathcal{M} \models \sigma$ if and only if $\mathcal{N} \models \sigma$.

   Hint: prove by induction a more general statement about arbitrary $\mathcal{L}$-formulas.

8. Let $\mathcal{L} := \{+, \cdot, 0, 1\}$ be the language of rings (i.e. $+$ and $\cdot$ are binary function symbols, and 0 and 1 are constant symbols).

   (a) Write down an $\mathcal{L}$-sentence σ_{field} whose models are exactly the fields.

   (b) For p a prime and for $p = 0$, give a set of formulas Φ_p whose models are exactly the fields of characteristic p.

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Section C

9. The *basic open subsets* (for the product topology) of the space of functions

$$2^{\mathbb{N}} = \{f : \mathbb{N} \to \{0,1\}\}$$

are the sets of functions which restrict to a given function on a finite subset, i.e. the sets

$$U_{A,g} := \{f : f|_A = g\} \subseteq 2^{\mathbb{N}}$$

where A is a finite subset of $\mathbb{N}$ and $g : A \to \{0,1\}$, and where $f|_A$ denotes the restriction of f to A.

By associating each $U_{A,g}$ to a suitable propositional formula, use the Compactness Theorem for propositional logic to show that if $\mathcal{U}$ is a set of basic open subsets of $2^{\mathbb{N}}$ such that $\bigcup \mathcal{U} = 2^{\mathbb{N}}$ (where $\bigcup \mathcal{U} = \{f \in 2^{\mathbb{N}} : \exists U \in \mathcal{U}\ f \in U\}$ denotes the union of the sets in $\mathcal{U}$), then there is a finite subset $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\bigcup \mathcal{U}_0 = 2^{\mathbb{N}}$.

In topological terminology, this precisely means that the product topology on $2^{\mathbb{N}}$ is compact. Show furthermore that, conversely, one can deduce propositional compactness from compactness of $2^{\mathbb{N}}$.

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