Mathematical Physiology

Weeks 5 – 8, Michaelmas Term 2025

Carles Falcó
falcoigandia@maths.ox.ac.uk
Office: S4.23

4.2 wave propagation

two pool model:
$$\dot{c} = r - kc - F$$
$$\dot{c}_s = F$$

intracellular matrix fluid

+ diffusion of sarcoplasm calcium

$$\partial_t c = r - kc - F + D\partial_x^2 c$$
$$\partial_t c_s = F$$

upon nondimensionalisation $t = \tilde{t}/k$
$$x = \ell \tilde{x}$$

and the lengthscale ℓ is fixed by diffusion:

$$D\partial_x^2 c = D\ell^{-2}\partial_{\tilde{x}}^2 c$$
$$\partial_t c = k\,\partial_{\tilde{t}} c$$

in particular, space gives one additional degree of freedom so we can set ℓ such that:

$$D\ell^{-2}k^{-1} = 1$$
$$\ell = \sqrt{D/k}$$ diffusion-to-decay lengthscale

hence, the model becomes:

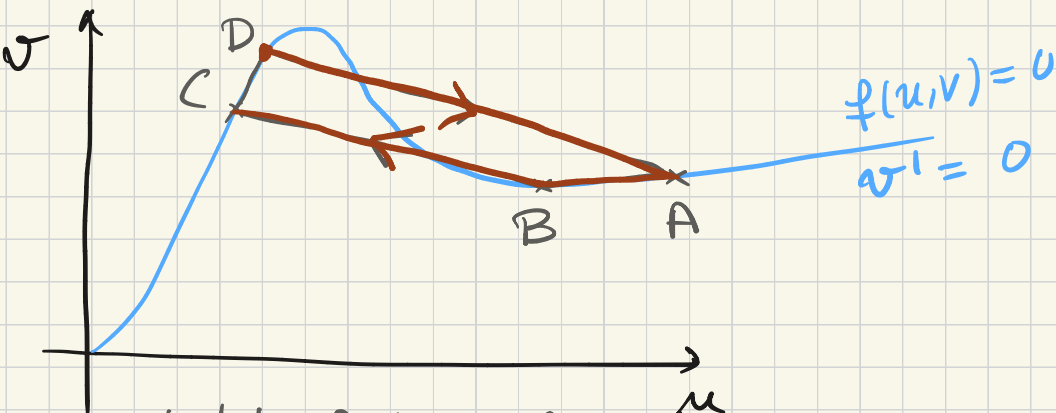
$$\begin{cases} (u+\gamma v)_t = \mu - u + u_{xx} \\ \varepsilon v_t = f(u,v) \end{cases}$$

next we look for travelling wave solutions:

$u = u(\xi)$
$v = v(\xi)$ where $\xi = x + \sigma t$

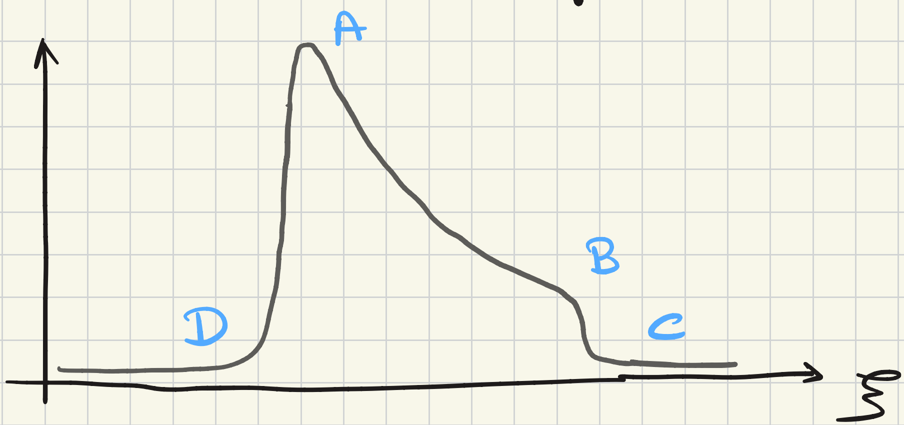
in TW coordinates $(\ )' = \dfrac{d}{d\xi}$

$$\begin{cases} \sigma(u+\gamma v)' = \mu - u + u'' \\ \varepsilon \sigma v' = f(u,v) \end{cases}$$



sketch of the wave in (u,v)-plane

the wave would look something like



recall $\varepsilon \sim k$, so smaller ε means slower decay of sarcoplasm concentration, which will lead to a faster wave. $\varepsilon \downarrow \Rightarrow \sigma \uparrow$

D → A Wave speed selection

this occurs over a thin region : $\xi = \varepsilon^a \eta$

$$\begin{cases} \sigma \varepsilon^{-a} (u + \gamma v)' = \mu - u + \varepsilon^{-2a} u'' \\ \sigma \varepsilon^{1-a} v' = f(u, v) \end{cases}$$

, (we still need to find $\sigma = \sigma(\varepsilon)$!)

$$\begin{cases} \sigma \varepsilon^{1-a} (u + \gamma v)' = \varepsilon (\mu - u) + \varepsilon^{1-2a} u'' \\ \sigma \varepsilon^{1-a} v' = f(u, V) \end{cases}$$

these can be balanced if $\boxed{a = \frac{1}{2}, \ \sigma = \frac{s}{\sqrt{\varepsilon}}}$

(note $\sigma \sim \frac{1}{\sqrt{\varepsilon}}$!)

to give: $\begin{cases} s(u+\gamma v)' = \varepsilon(\mu - u) + u'' \\ s v' = f(u,v) \end{cases}$

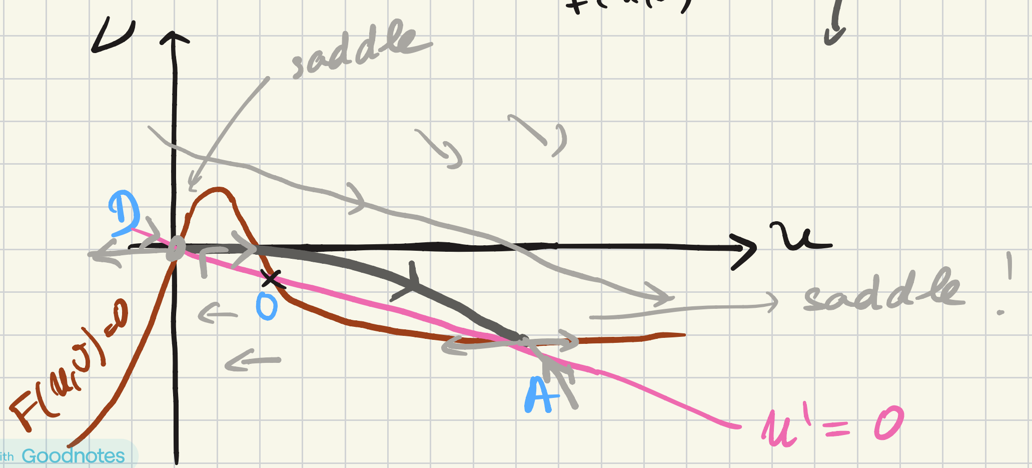
which at zeroth order in ε, gives:

$$\begin{cases} s(u+\gamma v)' = u'' \\ s v' = f(u,v) \end{cases} \qquad O(1)$$

then: $s(u+\gamma v) - s(u_D + \gamma v_D) = u' - \cancel{u_D'}^{\,0}$

now simply define $\begin{matrix} \mathcal{U} = u - u_D \\ \mathcal{V} = v - v_D \end{matrix}$ to get:

$$\left.\begin{cases} \mathcal{U}' = s(\mathcal{U} + \gamma \mathcal{V}) \\ \mathcal{V}' = \underset{= f(u,v)}{F(\mathcal{U},\mathcal{V})} \end{cases}\right\} \text{2D system!}$$



does a heteroclinic sol. from D to A exist? (saddle $\longrightarrow$ saddle)

only for a specific value of s!

assuming monotonicity of V:

$$\frac{dv}{du} = \frac{1}{s^2}\frac{F(u,v)}{u+\gamma v}$$

near $u=0$, and on the stable saddle separatrix $v \sim ku$ where k satisfies!

$$k = \lim_{u\to 0}\frac{dv}{du}$$

$$= \frac{1}{s^2}\lim_{u\to 0}\frac{F(0)+F_u(0)u+F_v(0)ku+O(u^2)}{u+\gamma k u}$$

$$= \frac{1}{s^2}\frac{F_u+F_v k}{1+\gamma k}$$

$F_v(0)<0$ so:

$$s^2 k(1+\gamma k) = F_u - |F_v| k$$

$$\boxed{s^2\gamma k^2 + k(s^2+|F_v|) - F_u = 0}$$

determinant $= \Delta = [s^2 + |F_v|]^2 + 4 F_u \gamma s^2$

which is $> 0 \Rightarrow$ one positive root

$$k = \frac{1}{2\gamma s^2}\left[-(s^2+|F_v|) + \sqrt{\Delta}\right]$$

$$\xrightarrow[s \downarrow 0]{} \frac{F_u}{|F_v|}$$ → motion along V-nullcline until O (stable node)

$$\xrightarrow[s \uparrow +\infty]{} 0$$ → $V \sim 0$

formally $s = +\infty$

$\Rightarrow \frac{dV}{dU} = 0 \Rightarrow V = 0$

so by continuity $\exists s > 0$ s.t $\exists$ heteroclinic orbit

is it unique? if trajectories variations were monotone, then yes! (see lecture notes)

$\Rightarrow$ s (and hence σ) is selected by the front!

end of $D \to A$

$A \to B$

this a slow phase, recall we had:

$$\begin{cases} \sigma(u+\gamma v)' = \mu - u + u'' \\ \sigma \varepsilon v' = f(u,v) \end{cases}$$

recall!

with $\sigma = \frac{s}{\sqrt{\varepsilon}}$

$D \to A$ occured on $\xi \sim \sqrt{\varepsilon}$

$\sqrt{\varepsilon}$? $\sim \varepsilon^b$? $b \geq 0$

what about $A \to B$? here $v' \sim 0$

and we look for possible balances

$$\xi = \varepsilon^{-b} \zeta$$

$$\begin{cases} s\,\varepsilon^{-1/2+b}(u+\gamma v)' = \mu - u + \varepsilon^{2b} u'' \\ s\,\varepsilon^{1/2+b} v' = f(u,v) \end{cases}$$

which can be balanced if $\boxed{b = \frac{1}{2}}$

then the wave has a width $\sim \frac{1}{\sqrt{\varepsilon}} + O(\sqrt{\varepsilon})$

and we obtain:

$$\begin{cases} s(u+\gamma v)' = \mu - u + \varepsilon u'' \\ s\varepsilon v' = f(u,v) \end{cases}$$

so at zeroth order in ε:

$$O(1) \begin{cases} f(u,v) = 0 \Rightarrow v = g(u) \\ u' s(1+\gamma g'(u)) = \mu - u \end{cases}$$

$$s u' = \frac{\mu - u}{1 + \gamma g'(u)}$$

$u > \mu$ and $g' > 0$ in this region so:

$$u' < 0$$

and u decreases from u_A to u_B

$B \to C$

this region is similar to $A \to D$, and also has width $\sqrt{\varepsilon}$, to order $O(1)$ in ε we get:

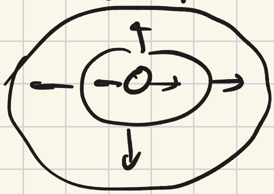
$$\begin{cases} u' = s\left[u - u_B + \gamma(v - v_B)\right] \\ s v' = f(u, v) \end{cases}$$

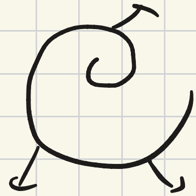
u_B is determined by a similar argument to s, connecting $B \to C$. See lecture notes for more details.

NOTE: The nondimensional diffusion coefficient in the lecture notes is $\nu = \varepsilon$. I've chosen this value to 1, and discussed the dependance of each region's width, and of the speed on ε. These two approaches are equivalent, with $s = \sqrt{\varepsilon}\,\sigma = O(1)$.

Waves in higher dimensions

two main types of wave can occur : target pattens & spiral waves





common framework:

$$u(\vec{x}, t) = u(r, \theta, t)$$

$$= u\big(\Omega t + m\theta - \psi(r)\big)$$

$m = 0$, target pattern $\rightarrow \psi' > 0$, $\Omega > 0$

$m \neq 0$, spiral wave

$$\downarrow$$

$$m\theta = \psi(r)$$

spiral waves have been observed in frog developing eggs.

5. the heart (electrochemical action)

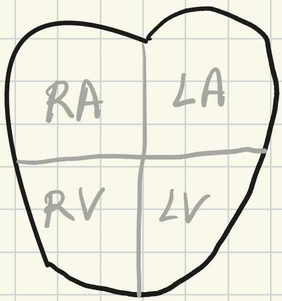
two main parts to the heart function:

(1) electrochemical action: creates muscle contraction to pump blood around the body

(2) mechanical action: enables unidirectional circulation via a system of valves

we begin studying (1)

the heart has four chambers



RA: right atrium
LA: left atrium
RV: right ventricle
LV: left ventricle

- blood flows into the RA from the venous system (superior and inferior vena cava), then flows to the RV and from there perfuses through the lungs to gain oxygen.

- blood returns to the heart into the LA, then to the LV and the LV pumps oxygenated blood out to the body through the aorta.

- Cardiac cells are electrically active:

→ Sino-atrial node cells, located at the RA, act as pacemakers with a periodic action potential.

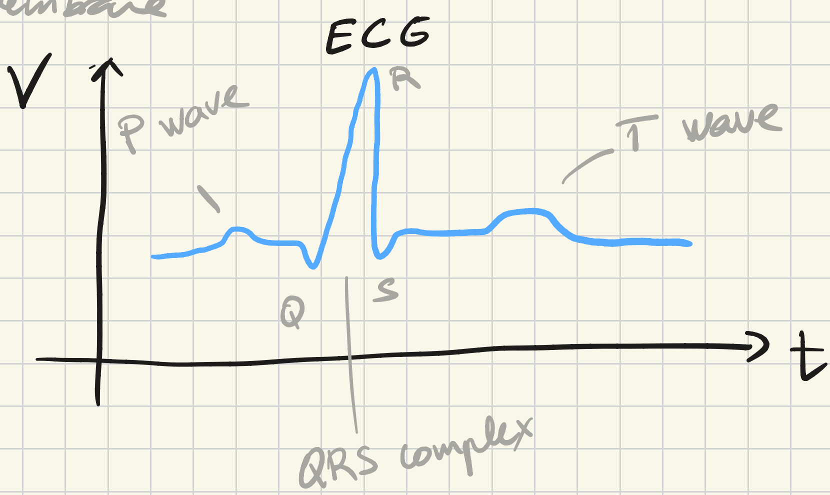
→ there are also other types of cells that are excitable (atrio-ventricular node cells, ventricular myocytes, Purkinje fibres) but with a different action potential.

→ compose the muscle of the ventricles
the ventricles contract when all the myocytes contract simultaneously.

electrocardiogram (ECG): body surface measurement of the electrical signals that stimulate the contraction of cardiac cell membranes and cause the depolarisation of the membrane potential, and the subsequent repolarisation.

(cell membranes are electrically polarised)

- different electrical signals can cause the membrane to depolarise. ECG measures these
- a second electrical signal repolarises the membrane



P = depolarisation of the atria

QRS = depolarisation of the ventricles

T = repolarisation of the ventricles

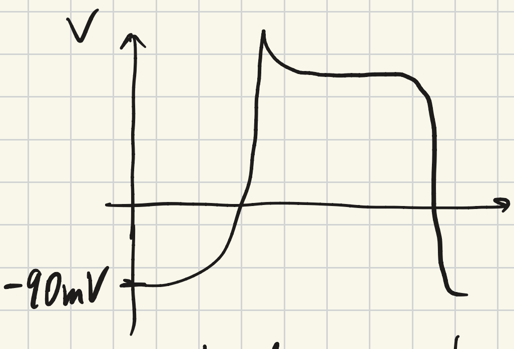
there are ~20 waves propagating through the heart from the sino-atrial node (RA).

the heart is not 1d! blockage of conduction paths can lead to "re-entrant" spiral waves, which cycle around the diseased tissue

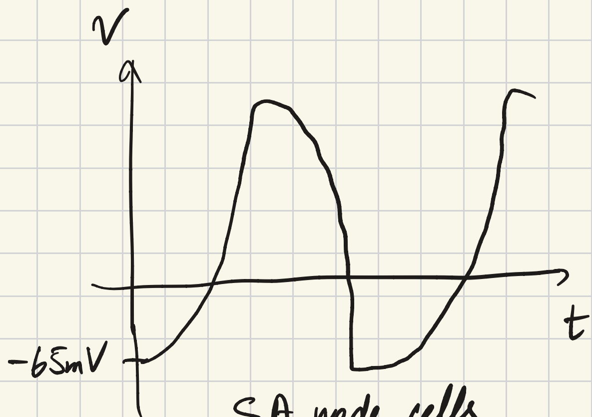
this can cause ventricular tachycardia

or if they become chaotic ventricular fibrillation

our goal: study the action potential of cardiac cells. in particular: ventricular myocytes and SA node cells.



ventricular myocytes



SA node cells

The Noble model (1962)

- Early model for the action potential of ventricular myocytes
- similar to Hodgkin - Huxley but more variables due to the greater number of currents involved.

ionic current = Na + K + leakage currents

$$C_m \frac{dV}{dt} = -I_i$$

! Noble model is largely based on experimental results.

$$I_i = I_{Na} + I_K + I_L$$

$$I_{Na} = \Big[\underset{\text{residual}}{g_0} + \overbrace{g_{Na}\, m^3 h}^{HH} \Big] (V - V_{Na})$$

switching on gate

switching off gate

$$I_K = \underbrace{(\underset{\text{instant}}{f_K} + \underset{\text{long-lasting}}{g_K n^4})}_{\text{cond. of K}} (V - V_K)$$

$$I_L = g_L (V - V_L)$$

constant leakage conductivity.

in the Noble model, g_L is very small

in fact with $g_L = 0$, the model shows self-sustained periodic behaviour

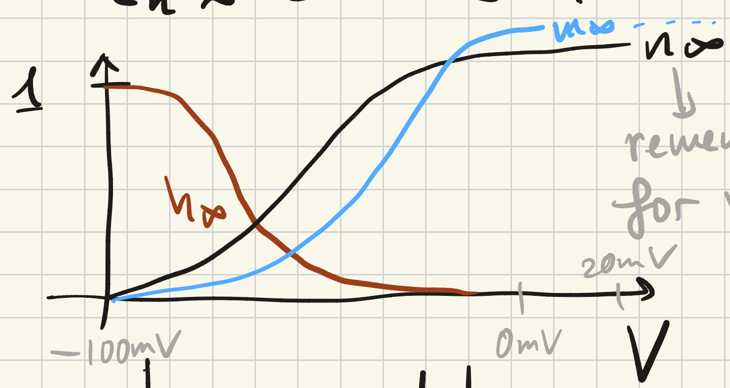
and the gate variables satisfy:

$$\tau_m \dot{m} = m_\infty - m,$$
$$\tau_h \dot{h} = h_\infty - h$$
$$\tau_n \dot{n} = n_\infty - n$$

where $\tau_m \sim 0.25\,ms$

$\tau_h \sim 8\,ms$

$\tau_n \sim 500\,ms$! much longer



remember graph for n_∞ !

experiments!

$m_\infty, h_\infty, n_\infty$ depend on V!

four eqs. to solve (V, n, m, h)

- to reduce the system we observe that $\tau_n >> \tau_h >> \tau_m$

↳ successive relaxation of the gates.

- we further assume $g_L \sim 0$ so that the model reduces to: (assuming $\tau_m \to 0$)

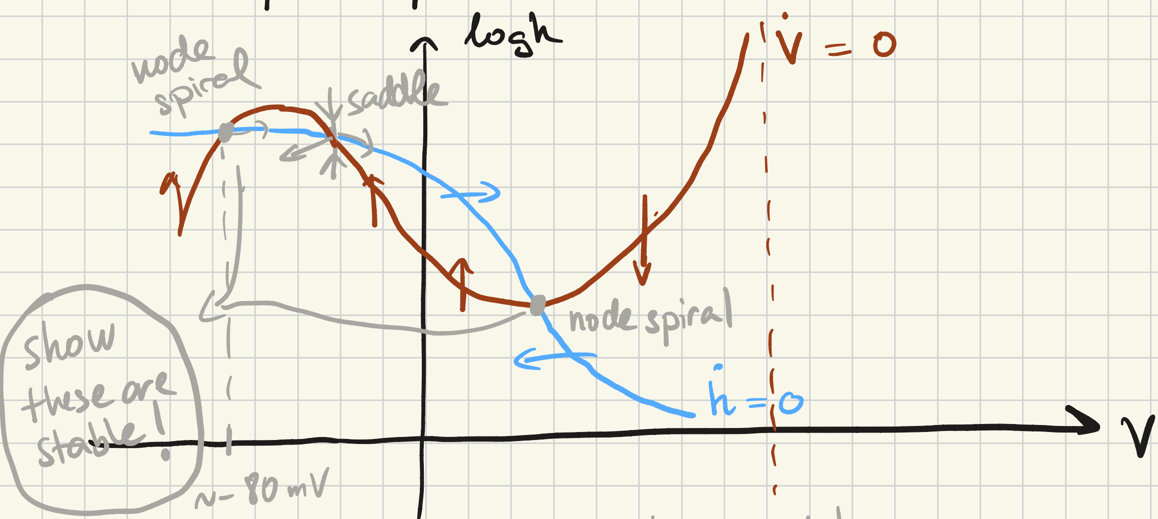
$$\begin{cases} \tau_h \dot{h} = h_\infty(V) - h \\ C_m \dot{V} = -\left[\left(g_0 + g_{Na} m_\infty^3(V) h\right)(V - V_{Na}) + f(V)(V - V_K) \right] \end{cases}$$

↳ fast phase of Noble model, timescale $\sim \tau_h$

where $f(V) = f_K(V) + g_K n^4$

on this phase $n \sim$ constant (recall $\tau_n \sim 500ms$)

then we could look in the phase plane for a trajectory connecting two fixed equations the phase plane looks like:



(or send me an email if you don't see it!)

more formally, we would rescale time by τ_h : $t = \tau_h \tilde{t}$ to get:

$$
\begin{cases}
\dfrac{\tau_m}{\tau_h}\dot m = m_\infty - m \,, & \dfrac{\tau_m}{\tau_h} << 1 \\
\dot h = h_\infty - h & \\
\dfrac{\tau_n}{\tau_h}\dot n = n_\infty - n & \dfrac{\tau_n}{\tau_h} = \dfrac{1}{\varepsilon} >> 1
\end{cases}
$$

$\hookrightarrow \dot n = \varepsilon(n_\infty - n)$

on the fast phase $n \approx n_\infty$

we also nondimensionalise $V \longrightarrow \bar{V}$

the *full* equation for $\dot{V}$ would become:

$$\dot{V} = -G(V, h, n)$$ (see Lecture Notes for full nondim.)

where:

$$G = -\big[\gamma_0 + \gamma_{Na}\, m_\infty^3(V)\, h\big](V_{Na} - V) + \phi\,(V+1) + \gamma_L\,(V + V_L)$$

1 1

$\phi = \phi_K(V) + \gamma_K n^4$

Noble took $\gamma_L = 0$ to begin with. (leakage part)

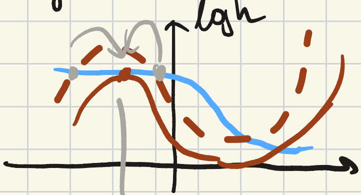
→ slow variations in n $\quad \dot{n} = \varepsilon(n_\infty - n)$

solutions starts at the left fixed point, where

V is low ⇒ n_∞ is low ⇒ n goes down

⇒ ϕ goes down ⇒ nullcline goes down

↓ nullcline moves down



the two steady states merge and

become unstable so we jump to the right steady state

from here, n_∞ is now high so $n_\infty - n > 0$
$\Rightarrow$ n goes up so the nullcline ($\dot{V}=0$) goes back up again until the left steady state reappears and we jump back there again.

[the two fixed points on the right hand side merge as well!)

* drawing from lecture!

waves in two or three dimensions

waves propagate through the heart in ~ two spatial dimensions

- heart muscle is composed of billions of interconnected VMs
- gap junctions form electrical connections between cells

in practice, homogenisation techniques allow us to derive continuum effective descriptions of electrical activity in the tissue.

we will consider the simplest possible model where $w \in \mathbb{R}^n$ is a vector of reactants

e.g. $w = (V, m, n, h)$

and the reaction kinetics are given by $f(w)$, where $f: \mathbb{R}^n \to \mathbb{R}^n$, and spatial transmission is modeled via linear diffusion:

$$\partial_t w = f(w) + \varepsilon \Delta w$$

- all reactants have same diffusivity: ε
- $f(w)$ is such that $\dot{w} = f(w)$ has a stable limit cycle behavior with period T → we call this sol. $w = W_0(t)$
- spatial transmission occurs on a slow timescale

the slow time is $\tau = \varepsilon t$

idea : because ε is small, different cells will oscillate with the same period, but with different phase. Also, spatial variations of the phase will evolve on a slow timescale ($\tau = \varepsilon t$)!

↳ method of multiple scales.

we seek a solution $w = w(x, t, \varepsilon t) = w(x, t, \tau)$

then the reaction-diffusion eq. becomes:

$$\partial_t w + \varepsilon \partial_\tau w = f(w) + \varepsilon \Delta w$$

we seek a perturbative sol: $w = w_0 + \varepsilon w_1 + \dots$

at order zero:

$$\partial_t w_0 = f(w_0)$$ limit cycle oscillations

$$w_0 = W_0(t + \psi(x, \tau))$$

period T !

slowly varying phase

this is a constant of integration

$$\int \frac{dw_0}{f(w_0)} = \int dt = t + C'(x, \tau)$$

at order $O(\varepsilon)$ we have:

$$\partial_t W_1 + \partial_\tau w_0 = Df(w_0)w_1 + \Delta w_0$$

$\Leftrightarrow$

$$\partial_t w_1 - Df(w_0)w_1 = -\partial_\tau w_0 + \Delta w_0$$

in terms of $w_0 = \overline{W}_0(t + \psi(x, \tau))$

we have:

$$J = Df(\overline{W}_0)$$

$$\partial_\tau w_0 = \partial_\tau \psi \, \overline{W}_0'$$

$$\Delta w_0 = \nabla \cdot [\nabla \psi \, W_0'] = \Delta \psi \, W_0' + |\nabla \psi|^2 W_0''$$

so putting everything together:

$$\partial_t w_1 - J w_1 = \left(-\partial_\tau \psi + \Delta \psi\right) W_0' + |\nabla \psi|^2 W_0''$$

this is a non-homogeneous equation!
(remember DEs II)

now use Fredholm's alternative:

u is a solution of $Lu = f$

if $\forall v \in \mathrm{null}(L)$ (i.e. $Lv = 0$) we have $\langle f, v \rangle = 0$

$$\int_\Omega f v = 0.$$

in our case we need the integral over one period $\int_0^T = 0$.

so, take v such that $\partial_t v - J v = 0$.

and then:

$$0 = \int_0^T v \left[W_0' (-\partial_z \psi + \Delta \psi) + W_0'' |\nabla \psi|^2 \right] dt$$

$$= (-\partial_z \psi + \Delta \psi) \int_0^T v W_0' \, dt + |\nabla \psi|^2 \int_0^T v W_0'' \, dt$$

so we obtain the solvability condition:

$$\left(\partial_\tau \psi - \Delta \psi\right)\int_0^T v W_0' = |\nabla \psi|^2 \int_0^T v W_0''$$

define: $\bar{\alpha} = \dfrac{\int_0^T v W_0'' \, dt}{\int_0^T v W_0' \, dt}$ so:

$$\partial_\tau \psi = \Delta \psi + \bar{\alpha} |\nabla \psi|^2$$

which gives the eq. for the phase

Summary: $\partial_t W = f(W) + \varepsilon \Delta W$.

to order zero we get:

$$W(x,t) \sim W_0\left(t + \psi(x,\tau)\right)$$

$$\tau = \varepsilon t$$

where

$$\partial_\tau \psi = \Delta \psi + \bar{\alpha} |\nabla \psi|^2$$

and $W_0(t)$ is the limit cycle of $\partial_t W = f(W)$.

some notes on: $\partial_z \psi = \Delta \psi + \bar{\alpha} |\nabla \psi|^2$

- both spiral and target patterns!
- integrated form of Burgers' equation

set $u = -\gamma^{-1} \nabla \psi$ and then

$$\partial_z (\bar{\nabla} \psi) = \nabla \Delta \psi + \bar{\alpha} \nabla |\bar{\nabla} \psi|^2$$

- note $\bar{\nabla} \cdot u = -\gamma^{-1} \Delta \psi$

so $\nabla (\nabla \cdot u) = -\gamma^{-1} \bar{\nabla} \Delta \psi$

and this is related to the vector Laplacian (note different from scalar Laplacian)

$$\Delta u = \bar{\nabla}(\bar{\nabla} \cdot u) - \bar{\nabla} \times (\bar{\nabla} \times u)$$

for $u \propto \bar{\nabla} \psi$ gradient field $\underline{\bar{\nabla} \times u = 0}$

so $\nabla (\nabla \cdot u) = \Delta u$

- also $\nabla |\nabla \psi|^2 = 2 D^2 \psi \nabla \psi$ → Hessian of ψ

$$= 2\gamma^2 (\nabla u) u$$

$$= 2\gamma^2 (u \cdot \nabla) u$$ (convective form)

so we get:

$$\gamma \partial_\tau u + 2\bar{\alpha}\gamma^2 (u\cdot\nabla)u = \gamma \Delta u$$

choose $\gamma = \dfrac{1}{2\bar{\alpha}}$

$$\partial_\tau u + (u\cdot\nabla)u = \Delta u .$$

viscous Burgers' equation, which is known to develop "shocks" (only for small diffusion)

$\Rightarrow$ jumps of the phase gradient

target patterns and spiral waves

we focus on $\partial_\tau \psi = \Delta\psi + \bar{\alpha}|\nabla\psi|^2$

relevant situation for the heart:

SA cell acting as a pacemaker located at the origin

Made with Goodnotes

these originate target patterns

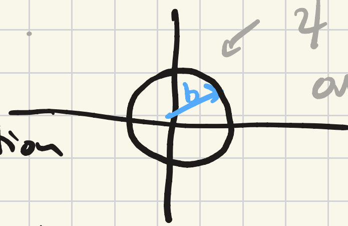
so assume (as a result of cells near ~0)

$$\psi(\tau, x) = \tau$$

$$\text{if} \quad |x| = r = b$$

$\uparrow$ radius in polar coordinates

there are BC's !



ψ is prescribed on $|x| = b$.

we look for a solution that decays radially:

$$\psi(\tau, x) = \psi(\tau, r) = \tau - f(r)$$

with $f(b) = 0$ and $\lim_{r \to +\infty} f(r) = 0$

we obtain

$$1 = -\frac{1}{r}(r f')' + \alpha (f')^2$$

$$\Leftrightarrow \quad f'' + \frac{1}{r} f' - \alpha (f')^2 + 1 = 0$$

this resembles some sort of Bessel eq.

but not quite yet!

Denote $g = e^{-\alpha f}$, then:

$$g' = -\alpha f' g$$

$$g'' = (\alpha^2 (f')^2 - \alpha f'')g$$

$$= \alpha g\left(\alpha (f')^2 - f''\right)$$

$$= \alpha g\left(\frac{1}{r} f' + 1\right)$$

$$= \alpha g - \frac{g'}{r}$$

$$\Rightarrow \quad g'' + \frac{g'}{r} - \alpha g = 0$$

and set $r = s/\sqrt{\alpha}$ so: $\alpha g'' + \frac{\alpha g'}{s} - \alpha g = 0$

$\Rightarrow g'' + \frac{1}{s} g' - g = 0$ Modified Bessel equation:

$$g(s) = A I_0(s) + B K_0(s)$$

$I_0(s) \to +\infty$ as $s \to +\infty$

hence $f \sim -\frac{1}{\alpha}\log(I_0)$ is not admissible

$A = 0$

and $f(r) = -\frac{1}{\alpha}\log\left[B K_0(\sqrt{\alpha}\, r)\right]$

$B = K_0(\sqrt{\alpha}\, b)^{-1}$ for BC at $r = b$.

$$\Rightarrow f(r) = -\frac{1}{\alpha}\log\left[\frac{K_0(\sqrt{\alpha}\, r)}{K_0(\sqrt{\alpha}\, b)}\right]$$

Q: what happens if $\alpha < 0$?

$$r \gg \frac{1}{\sqrt{\alpha}} \Rightarrow \psi \sim \tau - \frac{r}{\sqrt{\alpha}}$$

$$= -\frac{1}{\sqrt{\alpha}}\left(r - \sqrt{\alpha}\,\tau\right)$$

$\Rightarrow$ propagation speed $= \sqrt{\alpha}$

spiral waves also exist, but are associated with a core which allows the maintenance of a travelling wave circulating (this makes it slightly less exciting ?!)

assume BC at the core is given by:

$$r = b \Rightarrow \psi = \Omega z + m\theta$$

now use the ansatz:

$$\psi = \Omega z + m\theta - g(r)$$

to find the solution:

$$g = -\frac{1}{\alpha} \log\left[\frac{K_0(\sqrt{\alpha\Omega}\, r)}{K_0(\sqrt{\alpha\Omega}\, b)}\right]$$

so $\psi \sim \Omega z + m\theta - \sqrt{\frac{\Omega}{\alpha}}\, r$ as $r \to +\infty$.

this is an Archimedean spiral!

in general, these arise when electrical signals propagate unevenly through heart tissue due to heterogeneity.
(atrial or ventricular fibrilation)

curved front propagation

consider a wave in which the phase ψ of the potential varies rapidly within a thin region (wave front) which curves more slightly in other directions

we will work with the general equation:

$$\partial_t V = f(V) + \Delta V \quad (*)$$

assuming that it has a 1D travelling wave solution

$v = \mathcal{V}(\xi)$, $\xi = ct - x$, $c > 0$, $\mathcal{V}(\infty) = 1$, $\mathcal{V}(-\infty) = 0$

$$\Rightarrow \quad cV' = f(V) + V''$$

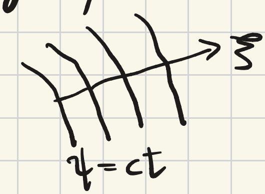
let ψ denote the phase of the wave and the location of the wave front is given as $\psi(x,t) = 0$

$x \in \mathbb{R}^d$, $d = 2,3$.

we want to descibe the evolution of the front, which is slowly varying in the directions transverse to the direction of propagation.

curvilineer coordinate, so that ξ measures distance along the normals of $\psi(x,t) = ct$

normel is $\quad \vec{n} = -\dfrac{\nabla\psi}{|\nabla\psi|}$



we look for a solution to $(*)$ of the form of $v = V(\psi(x,t))$ so that

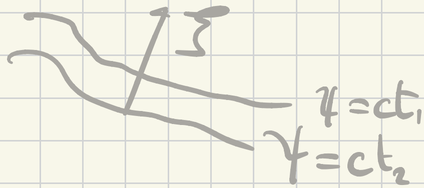
$$V'\partial_t \psi = \Delta\psi\, V' + |\nabla\psi|^2 V'' + f(V)$$

next we take a "quasi" 1D approximation assuming the front is thin

thus, in the curvilinear coordinate ξ we have:

$$\frac{\partial \bar{V}}{\partial \xi} = V'(\psi)\,|\nabla \psi|$$

this follows from:



$$\nabla V = V'(\psi)\,\nabla \psi$$

and $\frac{\partial V}{\partial \xi}$ is the derivative of V in the ξ direction

so $$\frac{\partial V}{\partial \xi} = \vec{n}\cdot\bar{\nabla} V = V'(\psi)\,\vec{n}\cdot\nabla\psi$$
$$= V'(\psi)\,|\nabla\psi|$$

directional derivative

then we also have

$$\partial_\xi^2 \bar{V} = V''(\psi)|\nabla\psi|^2 + \overbrace{\bar{V}'(\psi)}^{\frac{\partial_\xi V}{|\nabla\psi|}}\,\partial_\xi|\nabla\psi|$$

so we obtain

$$\frac{\partial_t \psi}{|\nabla\psi|}\,\partial_\xi \bar{V} = \frac{\Delta\psi}{|\nabla\psi|}\,\partial_\xi \bar{V} + \partial_\xi^2 \bar{V} - \frac{\partial_\xi V}{|\nabla\psi|}\,\partial_\xi|\nabla\psi| + f(\bar{V})$$

and we already know:

$$c\partial_\xi \bar{V} = f(V) + \partial_\xi^2 \bar{V}$$

so we obtain

$$\frac{\partial_t \psi}{|\nabla \psi|} = \frac{\Delta \psi}{|\nabla \psi|} + c - \frac{\partial_\xi |\nabla \psi|}{|\nabla \psi|}$$

or equivalently:

$$\partial_t \psi = \Delta \psi + c|\nabla \psi| - \partial_\xi |\nabla \psi|$$

remember $\partial_\xi \varphi = \nabla \varphi \cdot n$ for $\varphi = \varphi(x)$

so
$$\Delta \psi - \partial_\xi |\nabla \psi| = \nabla \cdot (\nabla \psi) - \nabla (|\nabla \psi|) \frac{\nabla \psi}{|\nabla \psi|}$$
$$= |\nabla \psi| \, \nabla \cdot \underbrace{\left(\frac{\nabla \psi}{|\nabla \psi|} \right)}_{"-\vec{n}}$$
$$= -|\nabla \psi| \, \nabla \cdot \vec{n}$$

so we could rewritte it as:

$$\underbrace{V_n := \frac{\partial_t \psi}{|\nabla \psi|}}_{\text{normal velocity of the surface}} = c - \nabla \cdot \vec{n}$$

eikonal equation

this equation relates v_n, the normal velocity of the surface, to the curvature $\nabla \cdot \vec{n}$

why is $\nabla \cdot \vec{n}$ related to the curvature ?

if ψ is flat (perfect front)

then $\vec{n} = \vec{ct}$ and $\nabla \cdot \vec{n} = 0$

if $\psi = ct$ looks like concentric rings of ~ circular shape

$$\vec{n} \sim \frac{(x,y)}{R}$$

$$\text{and} \quad \nabla \cdot \vec{n} = \frac{1}{R} = \text{curvature}$$

or in 3D:

$$\text{mean curvature} \sim \frac{\int_V \nabla \cdot \vec{n} \, dV}{V} = \frac{\int_{\partial V} dS}{V}$$

$$= \frac{S}{V} \sim \frac{R^2}{R^3} = \frac{1}{R}.$$