C4.1 Further Functional Analysis

Sheet 2 — MT 2025

For classes in week 5 or 6

This problem sheet is based on material up to and including Section 6 of the notes, together with appendix A.

Section A

1. Let X and Y be normed spaces and $T \in \mathcal{B}(X, Y)$
   (a) Show $(\text{ran } T)^\circ = \ker T^*$.
   (b) Use the Hahn-Banach theorem to show $(\text{ran } T^*)_\circ = \ker T$.

Solution:

(a)

$$(\text{ran } T)^\circ = \{f \in Y^* : f(Tx) = 0,\ x \in X\} = \{f \in Y^* : T^* f = 0\} = \ker T^*.$$

(b) If $x \in \operatorname{Ker} T$, then for any $f \in Y^*$, $(T^* f)(x) = f(Tx) = 0$, so $x \in (\text{ran } T^*)_\circ$. For the converse, if $x \in (\text{ran } T^*)_\circ$, then for all $f \in Y^*$, $(T^* f)(x) = f(Tx) = 0$. Hahn-Banach then implies that $x \in \operatorname{Ker} T$ (as if not $Tx \neq 0$, and so there would exist $f \in Y^*$ with $0 \neq f(Tx) = (T^* f)(x)$).

Jan Kristensen: kristens@maths.ox.ac.uk

2. Let X be a normed space.

   (a) Let $C \subset X$ be convex set. Show that the closure, $\overline{C}$ is convex.

   (b) Given a subset $A \subset X$, show that

$$\left\{ \sum_{i=1}^{n} \lambda_i a_i : n \in \mathbb{N},\ a_i \in A,\ \lambda_i \geq 0,\ \sum_i \lambda_i = 1 \right\}$$

   is the smallest convex subset of X containing A. This is known as the *convex hull* of A, and denoted $\mathrm{co}(A)$.

   (c) Given a subset $A \subset X$, show that $\overline{\mathrm{co}}(A)$ is the smallest closed convex subset of X containing A. This is known as the closed convex hull of A, denoted $\overline{\mathrm{co}}(A)$.

   (d) Use an example on the last sheet to provide closed convex sets $C_1, C_2 \subseteq X$ such that $\mathrm{co}(C_1 \cup C_2)$ is not closed.

   [You may find the closed convex hull construction useful in question B.4.]

Solution:

(a) Suppose $x, y \in \overline{C}$ and $0 < \lambda < 1$. Taking sequences $x_n \to x$ and $y_n \to y$ with $x_n, y_n \in C$, we have $\lambda x_n + (1-\lambda)y_n \in C$ and $\lambda x_n + (1-\lambda)y_n \to \lambda x + (1-\lambda)y$. Thus $\lambda x + (1-\lambda)y \in \overline{C}$ and so $\overline{C}$ is convex.

(b) It is easy to see that the given set is convex. Then show by induction that any convex set C has the property that $\sum_{i=1}^{n} \lambda_i c_i \in C$ whenever $c_1, \ldots, c_n \in C$ and $\lambda_i > 0$ have $\sum_{i=1}^{n} \lambda_i = 1$. Thus the displayed set is contained in any closed convex set containing A.

(c) In this case $\mathrm{co}(C_1 \cup C_2) = \{\lambda c_1 + (1-\lambda)c_2 : c_1 \in C_1,\ c_2 \in C_2,\ 0 \leq \lambda \leq 1\}$. Once one sees this closure follows exactly as in (a). To see this, note that the set on the right hand side is clearly contained in $\mathrm{co}(C_1 \cup C_2)$ from the expression of this set in (b); but it is also easily checked to be convex and contains $C_1 \cup C_2$.

(d) Sheet 1, B4 provides an example, with $C_1 = Y$ and $C_2 = Z$.

Jan Kristensen: kristens@maths.ox.ac.uk

Section B

1. Let X be a normed vector space and let Y be a subspace of X.

   (a) Suppose that Y is finite-dimensional. Show that Y is complemented in X, and that if Z is any closed subspace of X such that $X = Y \oplus Z$ algebraically, then X is in fact the topological direct sum of Y and Z.

   (b) What can you say if Y has finite codimension in X? [*Recall that the codimension of Y in X is the dimension of the quotient vector space X/Y.*]

Solution:

(a) Suppose that $Y = \operatorname{Span}\{x_1, \dots, x_n\}$ and let $\{g_1, \dots, g_n\} \subseteq Y'$ be the corresponding dual basis. Since Y is finite-dimensional the functionals g_k, $1 \leq k \leq n$, are continuous and by the Hahn-Banach Theorem we may find $f_1, \dots, f_n \in X^*$ such that $f_k|_Y = g_k$, $1 \leq k \leq n$. Define the map $P \colon X \to X$ by

$$Px = \sum_{k=1}^{n} f_k(x) x_k, \quad x \in X.$$

Then $P \in \mathcal{B}(X)$, $P^2 = P$ and $\operatorname{Ran} P = Y$, so Y is complemented in X.

Suppose that $X = Y \oplus Z$, where Z is closed, and let $P \colon X \to X$ denote the projection onto Y along Z. We may endow X/Z with the quotient norm and write $P = S \circ P_0 \circ \pi$, where $S \colon Y \to X$ is the embedding map, $P_0 \colon X/Z \to Y$ is defined by $P_0(x + Z) = Px$, $x \in X$, and $\pi \colon X \to X/Z$ is the canonical quotient operator. Since P_0 is injective the space X/Z must be finite-dimensional. Hence P_0 is bounded. Since S and π are also bounded, so is P. Hence $X = Y \oplus Z$ as a topological direct sum.

(b) Let $\{x_k + Y : 1 \leq k \leq n\}$ be a basis for X/Y and let $Z = \operatorname{Span}\{x_1, \dots, x_n\}$. Then $X = Y \oplus Z$. If Y is closed then by the previous part (with the roles of Y and Z reversed) the sum is topological, so Y is complemented in X. If Y is not closed then it cannot possibly be complemented. So Y is complemented if and only if it is closed. Moreover, if Y is closed and $X = Y \oplus Z$ algebraically for some subspace Z of X then Z is necessarily finite-dimensional and by the above argument the direct sum is topological.

Jan Kristensen: kristens@maths.ox.ac.uk

2. (a) Let X be a Banach space and suppose that $\{x_n : n \geq 1\}$ is a bounded subset of X. Show that there exists a unique operator $T \in \mathcal{B}(\ell^1, X)$ such that $Te_n = x_n$ for all $n \geq 1$ and $\|T\| = \sup_{n\geq 1} \|x_n\|$.

   (b) Prove that if X is a separable Banach space then $X \cong \ell^1/Y$ for some closed subspace Y of ℓ^1.

   (c) Deduce that ℓ^1 contains closed subspaces which are uncomplemented.

   [*You may assume that any closed infinite-dimensional subspace of ℓ^1 has non-separable dual. We might prove this at the end of the course.*]

Solution:

(a) Let $Y = c_{00} = \operatorname{Span}\{e_n : n \geq 1\}$, which is a dense subspace of ℓ^1. For $y = \lambda_1 e_1 + \ldots + \lambda_n e_n \in Y$ we define $Ty = \sum_{k=1}^n \lambda_k x_k$. If $C = \sup_{n\geq 1} \|x_n\|$ then

$$\|Ty\| \leq C \sum_{k=1}^{n} |\lambda_k| = C\|y\|_1.$$

Given $y \in \ell^1$ let $y_n \in Y$, $n \geq 1$, be such that $\|y - y_n\|_1 \to 0$ as $n \to \infty$. It follows from the last estimate that the sequence (Ty_n) is Cauchy and hence convergent, so there exists $x \in X$ such that $\|Ty_n - x\| \to 0$ as $n \to \infty$. Setting $Ty = x$ defines a linear operator $T\colon \ell^1 \to X$ satisfying $Te_n = x_n$, $n \geq 1$, and $\|Tx\| \leq C\|x\|$, $x \in X$. Thus $\|T\| \leq C$. Since $\|Te_n\| = \|x_n\|$, $n \geq 1$, we obtain $\|T\| \geq \|x_n\|$ for all $n \geq 1$ and hence $\|T\| \geq C$. To prove uniqueness suppose that $S \in \mathcal{B}(\ell^1, X)$ is such that $Se_n = x_n$, $n \geq 1$. Then S agrees with T on the dense subset Y of ℓ^1, so by continuity and density $S = T$.

(b) If X is a separable Banach space we may suppose that $\{x_n : n \geq 1\}$ is a dense subset of B_X. By part (a) there exists a (unique) map $T \in \mathcal{B}(\ell^1, X)$ of norm 1 such that $Te_n = x_n$, $n \geq 1$. Hence $\{x_n : n \geq 1\} \subseteq T(B_{\ell^1}) \subseteq B_X$, so the closure of $T(B_{\ell^1})$ coincides with B_X. It follows from the Successive Approximations Lemma that $T(B^\circ_{\ell^1}) = B^\circ_X$ and hence T is an isometric quotient operator, which is to say that $X \cong \ell^1/Y$ for $Y = \operatorname{Ker} T$.

(c) Let X be an infinite-dimensional separable Banach space with separable dual, for instance $X = \ell^p$ with $1 < p < \infty$. By part (b) we know that $X \cong \ell^1/Y$ for some closed subspace Y of ℓ^1. If Y is complemented in ℓ^1, then $\ell^1 = Y \oplus Z$ as a topological direct sum for some closed subspace Z of ℓ^1. By Sheet 1 Q8, we see that $Z \simeq \ell^1/Y \cong X$ and in particular Z is infinite-dimensional and has separable dual, which we are told is impossible. Hence Y is uncomplemented.

Jan Kristensen: kristens@maths.ox.ac.uk

3. (a) Let X be an infinite dimensional real normed space, and $f : X \to \mathbb{R}$ a linear functional. Show that if there is an open ball $B^0_X(x_0, r)$ such that $f(x) > 0$ for $x \in B^0_X(x_0, r)$, then f is continuous. Deduce that if f is unbounded, then $\ker f$ is dense in X.

   (b) Use the previous result to show that any infinite dimensional normed space X can be decomposed into a union $A \cup B$ of disjoint convex sets, with both A and B dense in X.

Solution:

(a) Suppose $f(x) > 0$ for $x \in B^0_X(x_0, r)$. Then for $\|x\| \leq 1$, $f(x_0 + rx) > 0$, so $f(x) > -f(x_0)/r$. By symmetry it follows that $|f(x)| \leq |f(x_0)|/r$, and so f is bounded. Now assume that f is unbounded. If $\ker f$ is not dense, then there is some open ball B on which f is not identically 0. But $f(B)$ is convex, so either $f(x) > 0$ for all $x \in B$, a contradiction, or $f(x) < 0$ for all $x \in B$, in which case $f(x) > 0$ for all $x \in -B$, again a contradiction.

(b) Regard X as a real vector space, and fix an unbounded real linear functional f_0 on X, and let $f(x) = f_0(x) - if_0(ix)$, so f is an unbounded linear functional on X. Then set $A = \{x \in X : \Re f(x) \leq 0\}$ and $B = \{x \in X : \Re f(x) > 0\}$. These are certainly disjoint convex sets. Then A contains the kernel of f_0 so is dense. For B, fix $x \in X$, and $\epsilon > 0$, and find some $y \in X$ with $f(y) = 0$ and $\|x - y\| < \epsilon$. Now take $x_0 \in X$ with $\Re f(x_0) > 0$, and so for $\delta > 0$ sufficiently small $\|x - (y + \delta x_0)\| < \epsilon$, and $y - \delta x_0 \in B$.

4. (a) Let C be a convex absorbing subset of a normed space. Show

$$\{x \in X : p_C(x) < 1\} \subseteq C \subseteq \{x \in X : p_C(x) \leq 1\},$$

   with equality in the first inclusion when C is open, and equality in the second when C is closed.

   (b) Let C be a convex balanced subset of a normed space, which contains a neighbourhood of 0 and is bounded. Show that p_C gives an equivalent norm on X.

   (c) Let Y be a subspace of a normed space $(X, \cdot)$, and let $|||\cdot|||$ be an equivalent norm on Y. Show that $|||\cdot|||$ can be extended to an equivalent norm on X.

Solution:

(a) If $p_C(x) < 1$, then there exists $0 < \lambda < 1$ with $\lambda^{-1}x \in C$. As $0 \in C$ (as a consequence of absorption), convexity gives $x \in C$. If $x \in C$, then $p_C(x) \leq 1$ by

Jan Kristensen: kristens@maths.ox.ac.uk

definition. If C is open, then for $x \in C$, there is a sufficiently small $\delta > 0$ such that $(1+\delta)x \in C$. Thus $p_C(x) < (1+\delta)^{-1} < 1$. When C is closed and $p_C(x) = 1$, there exists a sequence $\lambda_n \to 1$, such that $\lambda_n^{-1}x \in C$, so $x \in C$.

(b) It follows from results in lectures that p_C is a norm on X. Since C contains a neighbourhood, say $B_X^0(\epsilon)$ of 0, in X, for $\|x\| < \epsilon$, we have $x \in C$, so $p_C(x) \leq 1$. Thus $p_C(x) \leq \epsilon^{-1}\|x\|$. Suppose $K > 0$ is a bound for C. Then if $p_C(x) < 1$, then $x \in C$, so $\|x\| \leq K$. Thus $\|x\| \leq K^{-1}p_C(x)$.

(c) By rescaling $|||\cdot|||$ if necessary, we may assume that $|||y||| \leq \|y\|$ for all $y \in Y$. Let C be the convex hull of the union $B_{(X,\|\cdot\|)} \cup B_{(Y,|||\cdot|||)}$. Then C is a bounded convex set containing a neighbourhood of 0 in X, so p_C defines a norm on X equivalent to the original norm. Notice that (as Y is a subspace) $y \in Y$, $p_C(y) = p_{C\cap Y}(y)$, and since $B_{(X,\|\cdot\|)} \cap Y \subseteq B_{(Y,|||\cdot|||)}$, we have $C \cap Y = B_{(Y,|||\cdot|||)}$. Thus $p_C(y) \leq 1$ if and only if $|||y||| \leq 1$, i.e. $p_C(y) = |||y|||$ for $y \in Y$.

5. Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X,Y)$. Suppose there exists a constant $r > 0$ such that $\|T^*f\| \geq r\|f\|$ for all $f \in Y^*$.

   (a) Using the Hahn-Banach Separation Theorem, or otherwise, show that $B_Y(r)$ is contained in the closure of $T(B_X)$.

   (b) If X is complete, deduce that T is a quotient operator, and that T is an isometric quotient operator if T^* is an isometry.

   [This question is asking you to complete the missing bits from Theorem 5.16.]

Solution:

(a) Let C denote the closure of $T(B_X)$, and note that C is closed and convex. If the claim is false then there exists $y_0 \in B_Y(r) \setminus C$. By the Hahn-Banach Separation Theorem we may find a functional $f \in Y^*$ such that

$$\Re f(y_0) > \sup\{\Re f(y) : y \in C\} = \sup\{\Re f(Tx) : x \in B_X\} = \|T^*f\|.$$

But then $\|T^*f\| < \|f\|\|y_0\| \leq r\|f\|$, a contradiction. Thus $B_Y(r) \subseteq C$.

(b) If X is complete then the Successive Approximations Lemma shows that in fact $B_Y^\circ(r) \subseteq T(B_X^\circ)$ and hence T is a quotient operator. If T^* is an isometry then we may take $r = 1$ in the above arguments to obtain that $B_Y^\circ \subseteq T(B_X^\circ)$. But $\|T\| = \|T^*\| = 1$ and hence $T(B_X^\circ) \subseteq B_Y^\circ$. Thus $T(B_X^\circ) = B_Y^\circ$, and it follows that T is an isometric quotient operator.

6. Let $X = \ell^\infty$ and $Sx = (x_{n+1})$ for $x = (x_n) \in X$. Moreover, let $T = I - S$.

   (a) Show that $\operatorname{Ker} T = \{(\lambda, \lambda, \lambda, \dots) : \lambda \in \mathbb{F}\}$ and that $\operatorname{Ran} T \cap \operatorname{Ker} T = \{0\}$.

   (b) Let $Y = \operatorname{Ran} T \oplus \operatorname{Ker} T$ and let $P\colon Y \to Y$ be the projection onto $\operatorname{Ker} T$ along $\operatorname{Ran} T$. By considering the operators

   $$A_n = \frac{1}{n}\sum_{k=0}^{n-1} S^k, \quad n \geq 1,$$

   or otherwise, show that P is bounded and that $\|P\| = 1$.

   (c) Prove that there exists a functional $f \in X^*$ with $\|f\| = 1$ such that $f(Sx) = f(x)$ for all $x \in X$ and

   $$f(x) = \lim_{n\to\infty} x_n$$

   whenever $x = (x_n) \in c$.[1] Evaluate $f(x)$ when x is a periodic sequence.

Solution:

(a) It is clear that the space $\operatorname{Ker} T$ coincides with the space of fixed points of S, and each consists precisely of all constant sequences. Suppose that $x \in \operatorname{Ran} T \cap \operatorname{Ker} T$. Then $x = Ty = y - Sy$ for some $y \in X$ and hence $x_n = y_n - y_{n+1}$, $n \geq 1$. Since x is a constant sequence, with repeated entry $c \in \mathbb{F}$, say, we have $y_n = c + y_{n+1}$, $n \geq 1$. The only way this can happen for a bounded sequence y is if $c = 0$, which is to say that $x = 0$.

(b) Let $y \in Y$ and write $y = x + z$, where $x = Py \in \operatorname{Ker} T$ and $z = y - Py \in \operatorname{Ran} T$. Then $z = Tw = w - Sw$ for some $w \in X$. Now $A_n y = x + (w - S^n w)/n$ for $n \geq 1$, and hence $\|A_n y - Py\| \to 0$ as $n \to \infty$. Since $\|A_n\| \leq 1$, $n \geq 1$, we deduce that

$$\|Py\| = \lim_{n\to\infty} \|A_n y\| \leq \limsup_{n\to\infty} \|A_n\|\|y\| \leq \|y\|, \quad y \in Y,$$

so P is bounded with $\|P\| \leq 1$. Since $\|P\| \geq 1$ we have $\|P\| = 1$, as required.

(c) Let $g \in Y'$ be the functional defined by the relation $Py = g(y)e$, $y \in Y$, where $e = (1, 1, 1, \dots)$. Then $|g(y)| = \|Py\| \leq \|y\|$, $y \in Y$, so $g \in Y^*$ with $\|g\| = 1$, and by the Hahn-Banach Theorem there exists $f \in X^*$ such that $\|f\| = 1$ and $f|_Y = g$. In particular, $f(Tx) = 0$ and hence $f(Sx) = f(x)$ for all $x \in X$. Suppose that $x \in c$ and that $x_n \to L$ as $n \to \infty$. Then $\|S^n x - Le\| \to 0$ as $n \to \infty$. Since $f(x) = f(S^n x)$, $n \geq 0$, and $f(e) = 1$ we have

$$|f(x) - L| = |f(S^n x - Le)| \leq \|S^n x - Le\| \to 0, \quad n \to \infty,$$

[1] Recall that c is the subspace of ℓ^∞ consisting of convergent sequences.

Jan Kristensen: kristens@maths.ox.ac.uk

and hence $f(x) = L$. If $x \in X$ is periodic with period $p \geq 1$, then the sequence $y = (x + \cdots + S^{p-1}x)/p$ is constant with entry $c = (x_1 + \cdots + x_p)/p$, and hence

$$f(x) = \frac{1}{p}\sum_{n=0}^{p-1} f(S^n x) = f\left(\frac{1}{p}\sum_{n=0}^{p-1} S^n x\right) = f(y) = f(ce) = c.$$

7. Let X be a normed vector space and let Y be a subspace of X.
   (a) Writing $Y^{\circ\circ} = (Y^\circ)^\circ$ for the double annihilator of Y in X^{**}, show that there exists an isometric isomorphism $T\colon Y^{**} \to Y^{\circ\circ}$ such that $T \circ J_Y = J_X|_Y$. Deduce that Y is reflexive if and only if $Y^{\circ\circ} \subseteq J_X(Y)$.
   (b) Show that if X is reflexive and Y is closed, then both Y and X/Y are reflexive. [*Hint: Do the case of* Y *being reflexive first, and then consider* $(X/Y)^*$.]
   (c) Now suppose X is Banach and Y is a closed subspace such that both Y and X/Y are reflexive. Show that X is reflexive by following the outline:
      (i) Fix $\phi \in X^{**}$. Let $\pi\colon X \to X/Y$ denote the canonical quotient operator, and show that there exists $x \in X$ such that $J_{X/Y}(x + Y) = \phi \circ \pi^*$.
      (ii) Show that $\phi - J_X(x) \in Y^{\circ\circ}$, and use 7(b) to find $y \in Y$ for which $\phi = J_X(x+y)$.

Solution:

(a) Let $S\colon Y \to X$ be the embedding operator. Then S is an isometry, so S^* is an isometric quotient operator and therefore S^{**} is again an isometry. Since $S^*\colon X^* \to Y^*$ is the restriction operator given by $S^*f = f|_Y$, $f \in X^*$, we also have $\operatorname{Ran} S^{**} = (\operatorname{Ker} S^*)^\circ = Y^{\circ\circ}$. So we may define $T = S^{**}$ but with codomain $Y^{\circ\circ}$. Since $S^{**} \circ J_Y = J_X \circ S$ we have that $T \circ J_Y = J_X|_Y$, as required.

If Y is reflexive then given any $\phi \in Y^{\circ\circ}$, there exists $y \in Y$ with $J_Y(y) = T^{-1}(\phi)$ so $\phi = T(J_Y(y)) = J_X(y)$, i.e. $Y^{\circ\circ} \subseteq J_X(Y)$. Conversely, if $Y^{\circ\circ} \subseteq J_X(Y)$ and $\phi \in Y^{**}$, there exists $y \in Y$ with $T(\phi) = J_X(y)$. But $T(J_Y(y)) = J_X(y)$, so that as T is an isomorphism $\phi = J_X(y)$ and Y is reflexive,.

(b) Suppose that X is reflexive and let $\phi \in Y^{\circ\circ}$. Since $Y^{\circ\circ} \subseteq X^{**}$ there exists $x \in X$ such that $\phi = J_X(x)$. Moreover, $\phi(f) = f(x) = 0$ for all $f \in Y^\circ$ and hence, using the fact that Y is closed, we see that $x \in (Y^\circ)_\circ = Y$. Thus $Y^{\circ\circ} \subseteq J_X(Y)$, so Y is reflexive. Next we observe that $(X/Y)^* \cong Y^\circ$ and that Y° is a closed subspace of X^*. If X is reflexive then so is X^*, and by what we have just shown so must Y° be. Hence $(X/Y)^*$ is reflexive. But X is reflexive and therefore complete, so X/Y is also complete. It follows that X/Y is reflexive.

Jan Kristensen: kristens@maths.ox.ac.uk

(c) (i) As $\phi \circ \pi^* \in (X/Y)^{**}$ which is reflexive, there exists $x \in X$ such that $\phi \circ \pi^* = J_{X/Y}(x + Y)$.

(ii) We start by showing that $Y^\circ \subseteq \mathrm{Ran}(\pi^*)$. Indeed, for $f \in Y^\circ$, define $g \in (X/Y)^*$ by $g(x + Y) = f(x)$ (this is well defined and continuous by usual argument since π is quotient) and note that $\pi^* g = f$. This is the asserted inclusion. We continue with the sam notation and have then

$$\phi(f) = (\phi \circ (\pi^*)g) = g(x + Y) = f(x) = J_X(x)(f)$$

Therefore $\phi - J_X(x) \in Y^{\circ\circ}$. As noted in the first part of the proof of B.7(b), as Y is reflexive, we have $Y^{\circ\circ} \subseteq J_X(Y)$ so there exists $y \in Y$ with $\phi - J_X(x) = J_X(y)$, i.e. $\phi = J_X(x + y)$, and hence J_X is surjective and X is reflexive.

Jan Kristensen: kristens@maths.ox.ac.uk

Section C

1. (a) Let X be a normed vector space and let $P \in \mathcal{B}(X^{***})$ be given by $P = J_{X^*}J_X^*$. Show that P is the projection onto $J_{X^*}(X^*)$ along $J_X(X)^\circ$ and that $\|P\| = 1$.

   (b) (i) Show that if $T \in \mathcal{B}(\ell^\infty)$ with $\|T\| = 1$ and $Te_n = e_n$, $n \geq 1$, then $T = I$.

   (ii) Deduce that there does not exist a projection of norm 1 from ℓ^∞ onto c_0.

   (iii) Prove that there is no normed vector space X such that X^* is isometrically isomorphic to c_0.

Solution:

(a) For $f \in X^*$ we have

$$\big(J_X^*(J_{X^*}f)\big)(x) = (J_X x)(f) = f(x), \quad x \in X,$$

so $J_X^* J_{X^*}$ is the identity operator on X^*. Hence $P^2 = P$. Note also that

$$J_{X^*}(X^*) = P(J_{X^*}(X^*)) \subseteq \operatorname{Ran} P \subseteq J_{X^*}(X^*),$$

so $\operatorname{Ran} P = J_{X^*}(X^*)$. Since J_{X^*} is injective, $\operatorname{Ker} P = \operatorname{Ker} J_X^* = J_X(X)^\circ$. Moreover, $\|P\| \leq \|J_{X^*}\|\|J_X^*\| = 1$. Since $P \neq 0$ we must have $\|P\| = 1$.

(b) (i) Let $x \in \ell^\infty$ and let $y = Tx$. Fix $n \geq 1$. Then $\|x + \lambda e_n\|_\infty = |x_n + \lambda|$ and $\|y + \lambda e_n\|_\infty = |y_n + \lambda|$ for all $\lambda \in \mathbb{F}$ with $|\lambda|$ sufficiently large. Thus

$$|y_n + \lambda| = \|y + \lambda e_n\|_\infty = \|T(x + \lambda e_n)\|_\infty \leq |x_n + \lambda|$$

for $\lambda \in \mathbb{F}$ as above. It follows from elementary geometric considerations that $x_n = y_n$. Since $n \geq 1$ was arbitrary, we have $x = y$ and hence $T = I$.

(ii) Suppose that $P \in \mathcal{B}(\ell^\infty)$ has norm 1 and is such that $P^2 = P$ and $\operatorname{Ran} P = c_0$. Then P fixes elements of c_0 and in particular $Pe_n = e_n$, $n \geq 1$. By the previous result we see that $P = I$ contradicting the fact that $\operatorname{Ran} P \neq \ell^\infty$.

(iii) Let $Y = c_0$ and suppose that $S\colon X^* \to Y$ is an isometric isomorphism. Let $\Phi\colon \ell^\infty \to Y^{**}$ be the usual isometric isomorphism and let $P \in \mathcal{B}(X^{***})$ be as in part (a). Note that $J_Y \circ S = S^{**} \circ J_{X^*}$ and that S^{**} is an isometric isomorphism. Consider the operator $T \in \mathcal{B}(\ell^\infty)$ given by

$$T = \Phi^{-1} J_Y S J_{X^*}^{-1} P (S^{**})^{-1} \Phi,$$

Jan Kristensen: kristens@maths.ox.ac.uk

where we write $J_{X^*}^{-1}$ for the inverse of the map J_{X^*} with codomain $J_{X^*}(X^*)$. Then $\|T\| = 1$ and, using the fact that $J_Y(y) = \Phi(y)$ for $y \in c_0$, we have

$$Ty = \big(\Phi^{-1} J_Y S J_{X^*}^{-1} P (S^{**})^{-1} J_Y\big)(y) = \big(\Phi^{-1} J_Y S J_{X^*}^{-1} P J_{X^*} S^{-1}\big)(y) = y.$$

By part (b)(i) we have that $T = I$. In particular, J_Y must be surjective, which is a contradiction because c_0 is non-reflexive.

2. Given a normed vector space X, we say that X is injective [2] if whenever Y is a subspace of a normed vector space Z and $T \in \mathcal{B}(Y, X)$ there exists an operator $S \in \mathcal{B}(Z, X)$ such that $\|S\| = \|T\|$ and $S|_Y = T$.

(a) (i) Show that ℓ^∞ is injective

(ii) By proving first that any operator $T \in \mathcal{B}(\ell^\infty, c_0)$ such that $Te_n = e_n$, $n \geq 1$, must have norm $\|T\| \geq 2$, or otherwise, show that c_0 is not injective.

(iii) Is c_0 complemented in c, and if so what can you say about the norm of a complementing projection?

(b) Suppose that X is an injective normed vector space, and Y is a subspace of a normed vector space Z such that Y is isomorphic to X. Prove that Y is complemented in Z.

Solution:

(a) (i) Suppose that Y is a subspace of Z and that $T \in \mathcal{B}(Y, \ell^\infty)$. For $n \geq 1$ let $p_n \in (\ell^\infty)^*$ by given by $p_n(x) = x_n$ and let $g_n \in Y^*$ be given by $g_n = T^* p_n$. Thus $Ty = (g_n(y))$, $y \in Y$, and hence

$$\|T\| = \sup_{y \in B_Y} \sup_{n \geq 1} |g_n(y)| = \sup_{n \geq 1} \sup_{y \in B_Y} |g_n(y)| = \sup_{n \geq 1} \|g_n\|.$$

By the Hahn-Banach extension theorem there exist $f_n \in Z^*$, $n \geq 1$, such that $f_n|_Y = g_n$ and $\|f_n\| = \|g_n\|$ for all $n \geq 1$. Let $S \in \mathcal{B}(Z, \ell^\infty)$ be given by $Sz = (f_n(z))$, $z \in Z$. Then $S|_Y = T$ and moreover $\|S\| = \sup_{n \geq 1} \|f_n\| = \sup_{n \geq 1} \|g_n\| = \|T\|$, as required.

(ii) Suppose that $T \in \mathcal{B}(\ell^\infty, c_0)$ satisfies $Te_n = e_n$, $n \geq 1$, and that $\|T\| < 2$. Let $e = (1, 1, 1, \dots)$ and let $x = Te$. Then

$$|x_n - 2| \leq \|x - 2e_n\| = \|T(e - 2e_n)\| \leq \|T\| \|e - 2e_n\| \leq \|T\|, \quad n \geq 1,$$

[2] The terminology comes from category theory; X is an injective object in the category of normed spaces with contractive linear maps.

and hence $|x_n| \geq 2 - \|T\| > 0$, $n \geq 1$, contradicting the fact that $x \in c_0$. So if $T \in \mathcal{B}(\ell^\infty, c_0)$ and $Te_n = e_n$, $n \geq 1$, then $\|T\| \geq 2$. In particular, taking $Y = c_0$ and $Z = \ell^\infty$ it follows that the identity operator on Y has no norm-preserving extension to Z, so c_0 is not injective.

(iii) The same proof as above shows that any projection P of c onto c_0 must have $\|P\| \geq 2$. We assert that c_0 is complemented in c by a projection of norm exactly 2. Indeed, define $f\colon c \to \mathbb{F}$ by $f((x_n)) = \lim_{n\to\infty} x_n$, and then $P\colon c \to c$ by $P((x_n)_{n=1}^\infty) = (x_n - f(x))_{n=1}^\infty$. This is clearly a bounded linear operator of c (considered with the ℓ^∞ norm) and since also $P(e_n) = e_n$ for all $n \geq 1$ it is easy to see that it is a projection of c onto c_0. Note that P is the sum of id_c and $x \mapsto f(x)(1, 1, \dots)$, so it has norm at most 2. Hence $\|P\| = 2$.

(b) If $T \in \mathcal{B}(Y, X)$ is an isomorphism then we may find $S \in \mathcal{B}(Z, X)$ such that $S|_Y = T$. Consider the operator $P \in \mathcal{B}(Z)$ given by $Pz = T^{-1}Sz$, $z \in Z$. Then $P^2 = P$ and $\operatorname{Ran} P = Y$, so by a result from lectures Y is complemented in Z.

3. Let Y and Z be closed subspaces of a Banach space X and suppose that $X^* = Y^\circ \oplus Z^\circ$ as a topological direct sum. Show that $X = Y \oplus Z$ as a topological direct sum.

Solution: Not so much as a solution, as a link to some hints. This was on the 2020 exam paper as Q3(b), broken up there in to parts to get you going (the question had not appeared on a problem sheet in 2020). The last part though is still pretty tricky - see the solutions to the 2020 exam, which are available for current students through the institute website.

Jan Kristensen: kristens@maths.ox.ac.uk