B1.1 Logic

Sheet 4 — MT25

Except where otherwise stated, $\mathcal{L}$ denotes an arbitrary countable first-order language.

Section A

1. Suppose $(\exists x_i \phi \to \psi) \in \mathrm{Sent}(\mathcal{L})$. Show

$$(\exists x_i \phi \to \psi) \vdash \forall x_i (\phi \to \psi).$$

2. (a) Prove for any closed (i.e. variable-free) $\mathcal{L}$-terms t_1, t_2, t_3:

   (i) $\vdash (t_1 \doteq t_2 \to t_2 \doteq t_1)$.

   (ii) $\{t_1 \doteq t_2, t_2 \doteq t_3\} \vdash t_1 \doteq t_3$.

   (b) Prove that if $\Sigma \subseteq \mathrm{Sent}(\mathcal{L})$, and if t_i, t_i' are closed $\mathcal{L}$-terms[1] such that $\Sigma \vdash t_i \doteq t_i'$ for $i = 1, \ldots, k$, then:

   (i) If P is a k-ary relation symbol and $\Sigma \vdash P(t_1, \ldots, t_k)$, then $\Sigma \vdash P(t_1', \ldots, t_k')$.

   (ii) If f is a k-ary function symbol, then $\Sigma \vdash f(t_1, \ldots, t_k) \doteq f(t_1', \ldots, t_k')$.

[1] A term is *closed* if no variable appears in it.

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Section B

3. Suppose $P \in \mathcal{L}$ is a binary relation. Find an $\mathcal{L}$-formula which is in prenex normal form and is logically equivalent to

$$\forall x_0(\forall x_1 P(x_0, x_1) \to \forall x_1 \exists x_0 P(x_0, x_1)).$$

4. Do not use the Completeness Theorem for first-order logic when answering this question. You may use the Deduction Theorem and the fact that $\vdash \phi$ if ϕ is a tautology.

   (a) Let $\sigma \in \mathrm{Sent}(\mathcal{L})$ and $\phi \in \mathrm{Form}(\mathcal{L})$, and suppose $\mathrm{Free}(\phi) \subseteq \{x_i\}$. Show:

   (i) $\vdash (\exists x_i(\sigma \to \phi) \to (\sigma \to \exists x_i \phi))$.

   (ii) $\vdash ((\sigma \to \exists x_i \phi) \to \exists x_i(\sigma \to \phi))$.

   (b) Let $i, j \in \mathbb{N}$.

   (i) Suppose $\phi, \psi \in \mathrm{Form}(\mathcal{L})$ and $\mathrm{Free}(\phi) \subseteq \{x_i\}$ and $\mathrm{Free}(\psi) \subseteq \{x_i\}$.
   Show that $\vdash (\forall x_i(\phi \to \psi) \to (\forall x_i \phi \to \forall x_i \psi))$.

   (ii) Show that $\vdash (\forall x_i(\phi \to \psi) \to (\forall x_i \phi \to \forall x_i \psi))$ for any $\phi, \psi \in \mathrm{Form}(\mathcal{L})$.
   [Hint: use Lemma 9.12]

   (iii) Show $\vdash (\exists x_i \forall x_j \phi \to \forall x_j \exists x_i \phi)$ for any $\phi \in \mathrm{Form}(\mathcal{L})$.

5. (a) Prove the Compactness Theorem directly from the statements of the Completeness and Soundness Theorems and the fact that a proof in $S(\mathcal{L})$ uses only finitely many hypotheses.

   (b) For $n \in \mathbb{N}$ with $n \geq 2$, give a sentence τ_n such that for any $\mathcal{L}$, an $\mathcal{L}$-structure satisfies τ_n if and only if it has at least n elements.

   (c) Show that if a set of sentences Σ has no infinite model, then there is some $n \in \mathbb{N}$ such that every model of Σ has at most n elements.

6. Let $\mathcal{L} := \{+, \cdot, 0, 1\}$ be the language of rings, and recall from Sheet 3 that there exists a set of $\mathcal{L}$-formulas Φ_0 whose models are precisely the fields of characteristic 0. Using the Compactness theorem, prove that no single sentence σ_0 has this property.

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7. (a) Let σ_{field} be a sentence in the language $\mathcal{L} := \{+, \cdot, 0, 1\}$ whose models are precisely the fields. Recall that an ordered field is a field together with a linear order which is preserved by addition and by multiplication by positive elements. Write down a sentence τ in the language $\mathcal{L} := \{+, \cdot, <, 0, 1\}$ such that $\sigma := (\sigma_{\text{field}} \wedge \tau)$ axiomatises being an ordered field, i.e. such that the models of σ are precisely the ordered fields.

   (b) Consider the ordered fields $\mathbb{Q}$ and $\mathbb{R}$. Is $\{\sigma\}$ consistent? Is it complete?

   (c) Recall that the ordering on $\mathbb{R}$ is *Archimedean*, meaning that for every $x \in \mathbb{R}$ there is some $n \in \mathbb{N}$ with $-n < x < n$. Use the Compactness Theorem to prove that Archimedeanity is not a first-order property: that is, there is no set of $\mathcal{L}$-sentences Σ whose models are precisely the Archimedean ordered fields.

   [Hint: introduce a new constant symbol c.]

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Section C

8. Find a $\{+, \cdot, 0, 1\}$-sentence τ such that $\sigma_{\text{field}} \cup \tau$ is consistent, and every model is a field of characteristic 0. [Hint: Use the fact that in a finite field, every element is a sum of two squares.]

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