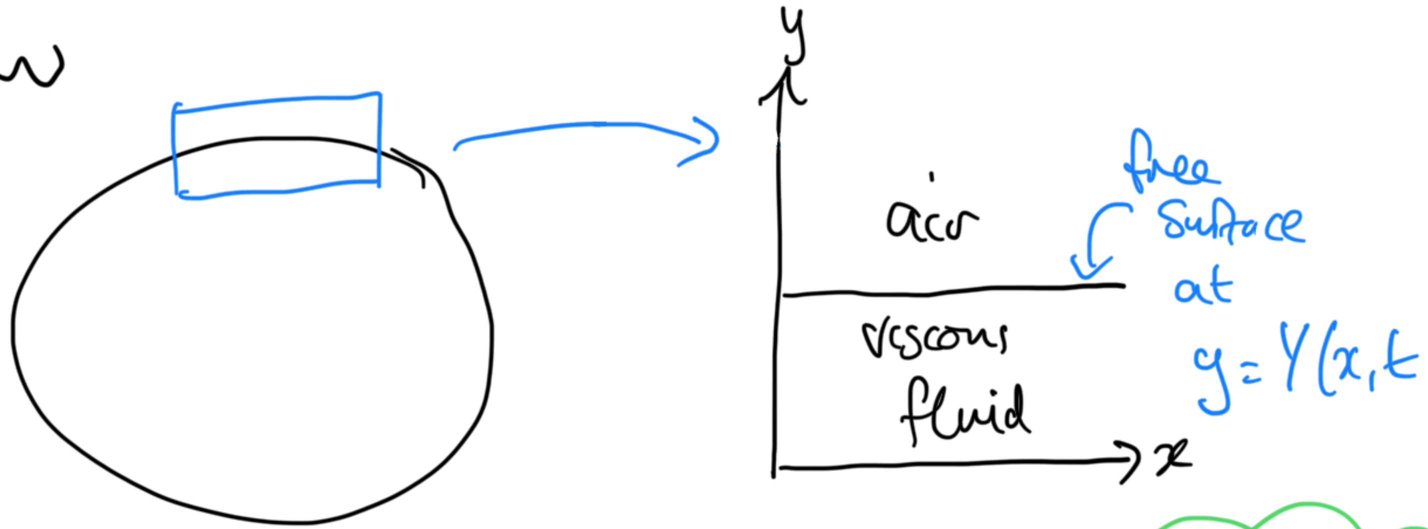
The Saffman-Taylor Instability

Zoom in to look at the local problem near the free boundary in the Hele-Shaw flow



The flow is governed by

Here I absorbed factor of 12 into p

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0, \quad \bar{u} = -\frac{\partial p}{\partial x}, \quad \bar{v} = -\frac{\partial p}{\partial y} \qquad (7.15)$$

At the free surface, we have

$$p = 0 \quad \text{and} \quad \bar{v} = \frac{\partial Y}{\partial t} + \bar{u}\frac{\partial Y}{\partial x} \quad \text{on } y = Y$$

This is the kinematic condition: equivalent to $\frac{dR}{dt} = \underline{\bar{u}} \cdot \underline{e_r}$

The "base" solution (with no instability) is a steady travelling wave solution with the free surface located at $y = Vt$ and

moving with speed V, in which
$u_0 = 0$, $\bar{v}_0 = V$, $Y_0 = Vt$ & $P_0 = -V(y - Vt)$.

We drive the flow by prescribing the pressure gradient

$$\frac{\partial P_0}{\partial y} \to -V \quad \text{as } y \to -\infty$$

this gives V.

Our aim is to consider small perturbations to this base state & see what happens.

The key step is to change to a travelling wave frame (ξ, η, τ) moving with the base state. We write $\xi = x$, $\eta = y - Vt$, $\tau = t$, and set

$$\bar{u} = u_0 + \varepsilon \tilde{u} = \varepsilon \hat{u}(\xi, \eta, \tau)$$

$$\bar{v} = v_0 + \varepsilon \tilde{v} = V + \varepsilon \hat{v}(\xi, \eta, \tau)$$

$$Y = Y_0 + \varepsilon \tilde{Y} = Vt + \varepsilon \tilde{Y}(\xi, \tau)$$
$$= V\tau + \varepsilon \hat{Y}(\xi, \tau)$$

$$\& \ P = P_0 + \varepsilon \tilde{P} = -V(y - Vt) + \varepsilon \hat{P}(\xi, \eta, \tau)$$
$$= -V\eta + \varepsilon \hat{P}(\xi, \eta, \tau)$$

where ε is the small parameter characterising the size of the initial perturbation.

Using the chain rule,

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau} - V\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}$$

and so (7.15) becomes

$$\boxed{\frac{\partial \tilde{u}}{\partial \xi} + \frac{\partial \tilde{v}}{\partial \eta} = 0, \quad \tilde{u} = -\frac{\partial \tilde{p}}{\partial \xi}, \quad \tilde{v} = -\frac{\partial \tilde{p}}{\partial y}}$$

in $\eta < Y - Vt = \varepsilon \tilde{Y}$.

Then, since $p=0$ at the free surface $\eta = \varepsilon \tilde{Y}$,

$$-V\varepsilon\tilde{Y} + \varepsilon \tilde{p}(\xi, \varepsilon\tilde{Y}, \tau) = 0$$

$$\Rightarrow \boxed{\tilde{p}(\xi, \varepsilon\tilde{Y}, \tau) = V\tilde{Y},}$$

while the kinematic condition becomes

$$V + \varepsilon\tilde{v} = V + \varepsilon\frac{\partial \tilde{Y}}{\partial \tau} + \varepsilon\tilde{u}.\varepsilon\frac{\partial \tilde{Y}}{\partial \xi}$$

ie $$\boxed{\tilde{v} = \frac{\partial \tilde{Y}}{\partial \tau} + \varepsilon\tilde{u}\frac{\partial \tilde{Y}}{\partial \xi} \text{ on } \eta = \varepsilon\tilde{Y}}.$$

We also set $\tilde{p} = o(-\eta)$ as $\eta \to -\infty$

means "asymptotically smaller than $|\eta|$"

We now follow a similar recipe to that used for water waves in F&W. We first linearise the problem (ie set $\varepsilon = 0$ & lose the blue terms above) to find:

$$\boxed{\begin{array}{l} \dfrac{\partial \tilde u}{\partial \xi} + \dfrac{\partial \tilde v}{\partial \eta} = 0, \quad \tilde u = -\dfrac{\partial \tilde p}{\partial \xi}, \quad \tilde v = -\dfrac{\partial \tilde p}{\partial \eta} \quad \text{in } \eta < 0 \\[2ex] \text{with} \quad \left.\begin{array}{l} \tilde p = V\tilde\psi \\ \tilde v = \dfrac{\partial \tilde \psi}{\partial \tau} \end{array}\right\} \text{ on } \eta = 0 \ \ \& \ \ p = o(-\eta) \text{ as } \eta \to -\infty \end{array}}$$

We seek a separable solution of the form

$$\tilde\psi = e^{ik\xi + \lambda\tau}, \quad p = g(\eta)\, e^{ikx + \lambda\tau}$$

wave number

complex growth rate

The form for p is motivated by $\tilde p = V\tilde\psi$ on $\eta = 0$

Since $\nabla^2 p = 0$, the problem becomes

$$g'' - k^2 g = 0 \qquad \text{in } \eta < 0$$

with
$$g(0) = V$$
$$g'(0) = -\lambda$$
$$g \text{ bdd as } \eta \to -\infty.$$

This is an eigenvalue problem for $\lambda(k)$.
The solution for g that has the right behavior as $\eta \to -\infty$ is

$$g = Ae^{|k|\eta}.$$

Applying the boundary conditions at $\eta = 0$, we have $A = V$ and $A|k| = -\lambda$.

$\therefore$ $$\boxed{\lambda = -V|k|}$$

Hence, perturbations decay ($\lambda<0$) if $V>0$ and the free boundary is stable, and perturbations grow ($\lambda>0$) if $V<0$ and the free boundary is unstable.

This is an example of the Saffman-Taylor instability: The interface between two immiscible fluids is <u>unstable</u> when it <u>moves towards</u> the more viscous fluid.

<u>NB</u> The growth rate of the instability is proportional to $|k|$. Shorter wavelength (larger $|k|$) disturbances grow faster than longer wavelength ones, and grow into "viscous fingers", whose evolution is highly nonlinear and very sensitive to the driving force, surface tension, roughness of substrate etc.