C5.6 Applied Complex Variables Plan

# Lectures	Subject	Problem Sheet
3	Revision of core analysis and conformal mapping	1
2	Schwarz-Christoffel formulae and BVPs via conformal mapping	1, 2
2	Steady free surface flows	2
1	Unsteady free surface flows	2

# lectures	Subject	Problem Sheets
3	Plemelj formulae, mixed BVPs; Cauchy singular integral equations	3
2	Complex and generalized Fourier transforms	3, 4
3	The Wiener-Hopf Method	4

Motivation: problems leading to Laplace's equation

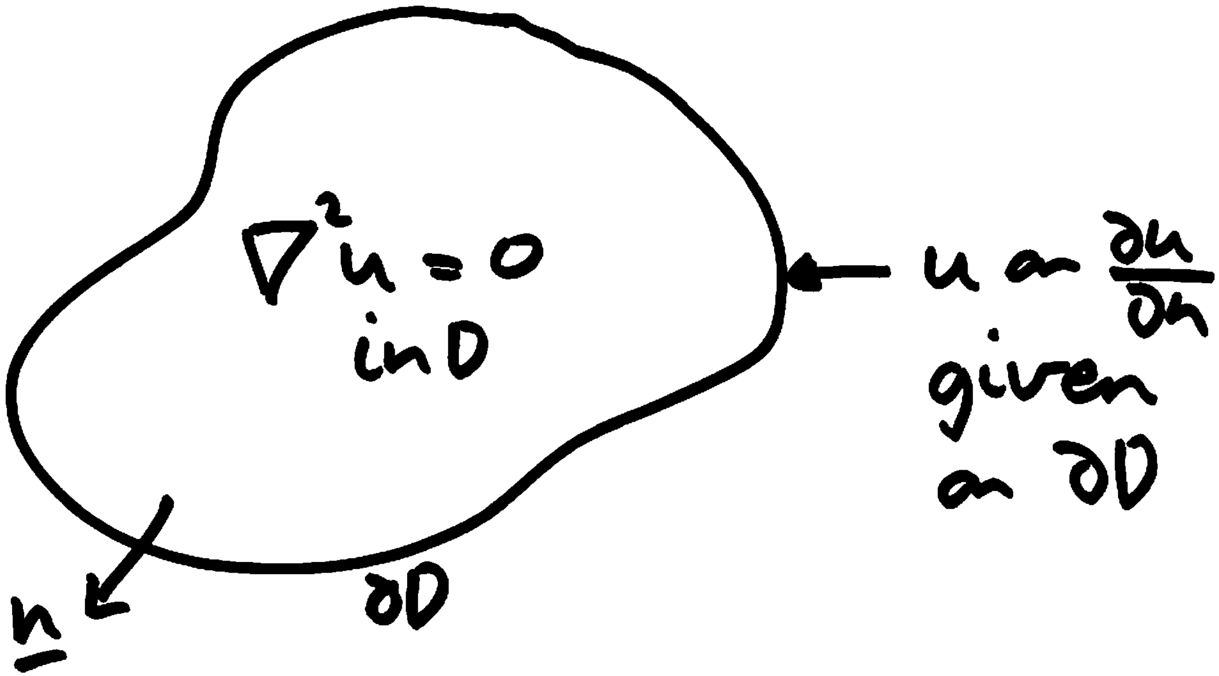
Steady heat flow

- Conservation of energy $\nabla \cdot \underline{q} = 0$

  Fourier's law $\underline{q} = -k \nabla u$

  $$\left.\begin{array}{l} \nabla \cdot \underline{q} = 0 \\ \underline{q} = -k\nabla u \end{array}\right] \Rightarrow \nabla^2 u = 0$$

- Typical BCs:



Inviscid fluid flow

- Steady, 2D, incompressible, irrotational

$$\Rightarrow \quad \underline{u}(x,y) = (u,v,0), \quad \nabla\cdot\underline{u} = 0, \quad \nabla\wedge\underline{u} = \underline{0}$$

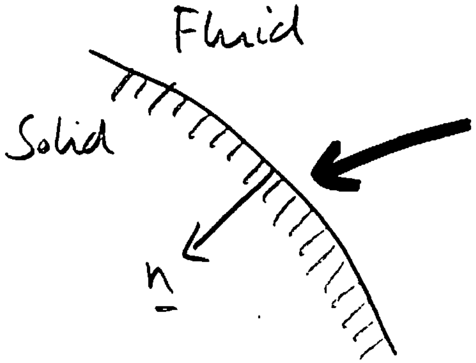
$$\Rightarrow \quad \exists\,\psi,\phi \text{ s.t. } \quad u = \psi_y = \phi_x, \quad v = -\psi_x = \phi_y$$

$$\Rightarrow \quad \nabla^2\psi = 0, \quad \nabla^2\phi = 0$$

- Pressure given by Bernoulli (no gravity):

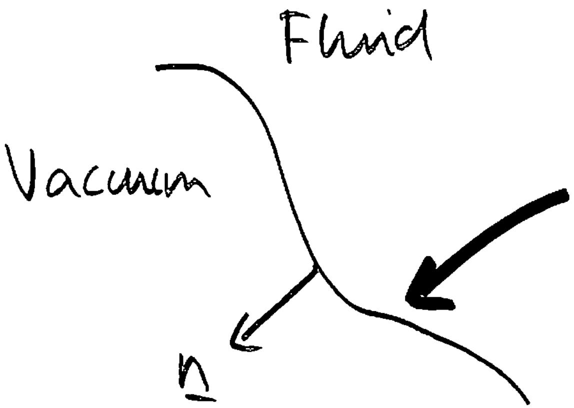
$$\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 = \text{constant}.$$

- BC at a solid surface:



$\underline{u}\cdot\underline{n}=0$ or $\dfrac{\partial\phi}{\partial n}=0$ or $\psi=\text{constant}$

- BCs at a free surface (no surface tension):



$\underline{u}\cdot\underline{n}=0$ and $p=\text{constant}$

$$\Downarrow$$

$|\underline{u}|^2=\text{constant}.$

- Others include:

  gravitation

  electromagnetism

  membranes

  linear elasticity

  Darcy flow

  etc

Review of core complex analysis

Notation

- $z = x + iy \in \mathbb{C}$ $(x, y \in \mathbb{R})$; $\bar{z} = x - iy$.

- D is a region (an open, path-connected subset of $\mathbb{C}$), simply-connected unless stated otherwise, with boundary ∂D.

- Γ is a contour (a piecewise, continuously differentiable, simple path in $\mathbb{C}$) oriented with the positive (anti-clockwise) orientation closed with interior $\mathrm{Int}(\Gamma)$ unless stated otherwise.

- $D(a, R) := \{z \in \mathbb{C} : |z - a| < R\}$, disc centre a, radius R.

Derivatives

- $f(z)$ holomorphic on D ($f \in H(D)$)

$$\Longleftrightarrow \quad f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} \text{ exists } \forall z \in D.$$

- Much more restrictive than real case $\because$ need

$$\lim_{h \to 0} = \lim_{\mathrm{Im}(h) \to 0} \ \lim_{\mathrm{Re}(h) \to 0} = \lim_{\mathrm{Re}(h) \to 0} \ \lim_{\mathrm{Im}(h) \to 0}$$

- Let $f(z) = u(x,y) + iv(x,y)$ $(u = \mathrm{Re}(f), v = \mathrm{Im}(f))$

$$\Rightarrow \quad f'(z) \underset{\substack{\uparrow \\ h \text{ real}}}{=} u_x + iv_x \underset{\substack{\uparrow \\ h \text{ pure imaginary}}}{=} \frac{u_y + iv_y}{i}$$

$\Rightarrow$ $u_x = v_y$, $u_y = -v_x$ <u>Cauchy-Riemann Equations</u>

$\Rightarrow$ $\nabla^2 u = 0$, $\nabla^2 v = 0$ <u>Laplace's equation</u>

NB: u, v called harmonic conjugates.

NB: Cauchy-Riemann Equations

$$\Leftrightarrow \quad \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) = 0$$

Chain rule with $z, \bar{z}$ independent.

Integrals

- **Cauchy's Theorem**

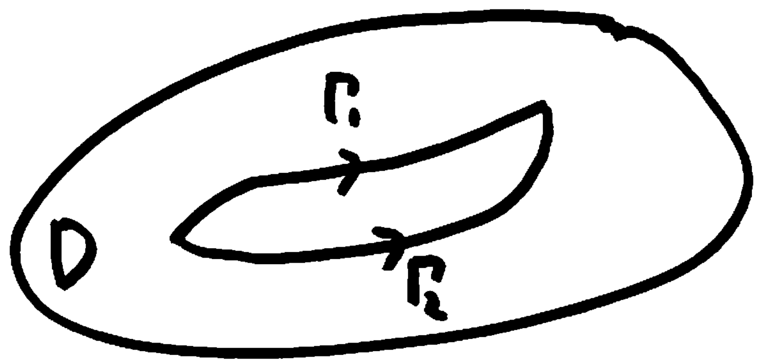
$$f \in H(\mathrm{Int}(\Gamma)) \cap C(\Gamma) \implies \oint_{\Gamma} f(z)\,dz = 0$$

f continuous on Γ

- **Deformation Theorem**

$f \in H(D)$, $\Gamma_1, \Gamma_2 \subseteq D$ open contours with same end points

$$\implies \int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz$$



$\Rightarrow$ Anti-derivative of $f \in H(D)$ well-defined by
$$F(z) = \int_{z_0}^{z} f(\hat{z})\, d\hat{z}, \quad z \in D, \; z_0 \in D \text{ fixed.}$$

- Can show $\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| \to 0$ as $h \to 0$

$\Rightarrow F' = f$ <u>Fundamental Theorem of Calculus</u>.

- <u>Cauchy's integral formula</u>: $f \in H(D)$, $\Gamma \subseteq D$ and $z \in \mathrm{Int}(\Gamma) \Rightarrow$ $$f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(t)}{t - z}\, dt$$

Pf: $$\mathrm{RHS} \underset{\substack{\uparrow \\ \text{Def}^{\text{n}} \text{ thm}}}{=} \frac{1}{2\pi i} \int_{|t-z|=\varepsilon} \frac{f(z)}{t-z} + \frac{f(t)-f(z)}{t-z}\, dt \to f(z) + 0$$
as $\varepsilon \to 0$ by cty of f.

- Can show $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = \frac{1}{2\pi i} \oint_\Gamma \frac{f(t)}{(t-z)^2} dt$

- Induction $\Rightarrow$ $f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(t)}{(t-z)^{n+1}} dt$

  $\Rightarrow$ infinite differentiability!

  $\Rightarrow$ focus of complex analysis is on singularities.

- <u>Liouville's Theorem</u>: Any bounded entire (i.e. $f \in H(\mathbb{C})$) function is constant.

Pf: $f'(z) = \frac{1}{2\pi i} \oint_{|t-z|=R} \frac{f(t)}{(t-z)^2} dt \to 0$ as $R \to \infty$.

• <u>Corollary</u>: $f \in H(\mathbb{C})$ and $f(z) = O(z^n)$ as $|z| \to \infty$ $(n \in \mathbb{N})$

$\Rightarrow f$ is a polynomial of degree n.

Pf: Apply Liouville to $f^{(n)}(z)$.

• <u>Taylor's Theorem</u>: $f \in H(D(a,R))$

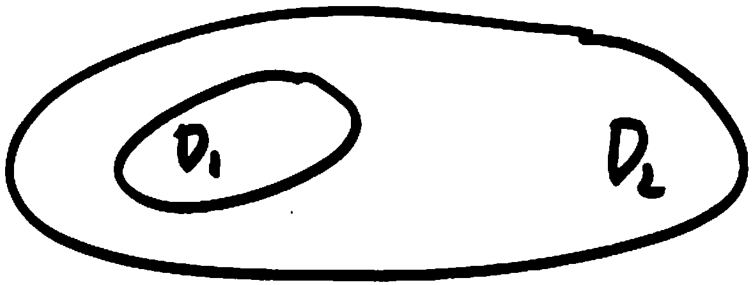
$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \quad \text{for } z \in D(a,R).$$

NB: Radius of convergence (i.e. maximum possible R) is distance from $z = a$ to nearest singularity,

e.g. $$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \quad \text{for } |z| < 1.$$

- <u>Analytic continuation</u> (AC) is the process of extending the domain of definition of a holomorphic function.

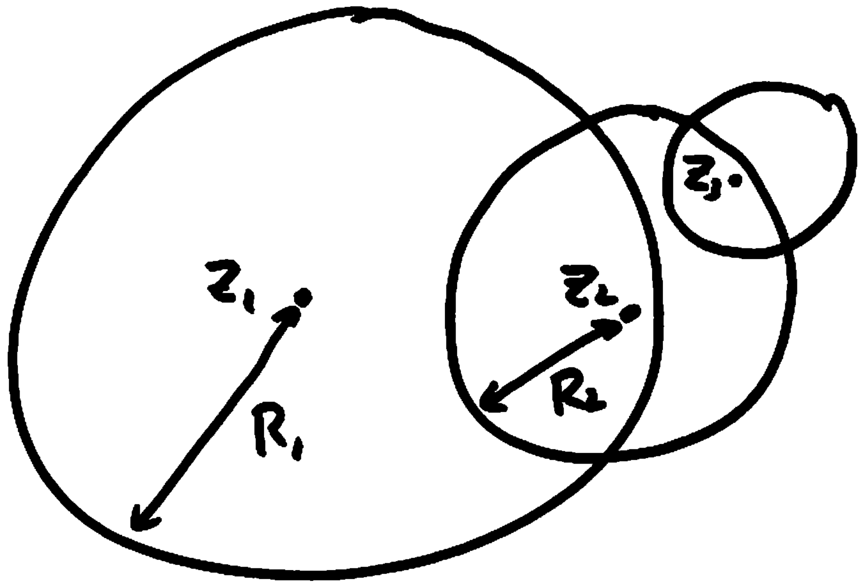
- If $f_j \in H(D_j)$ with $f_1 = f_2$ on $D_1 \subseteq D_2$, then f_2 is an AC of f_1.



- E.g. $f_1(z) = \sum_{n=0}^{\infty} z^n \in H(D(0,1))$ has AC

$$f_2(z) = \frac{1}{1-z} \in H(\mathbb{C} \setminus \{1\}).$$

- AC possible via Taylor series using <u>circle-chain method:</u>



$$f \in H(D(z_1, R_1)) \cup H(D(z_2, R_2)) \cup \dots$$

Construct via Taylor & convergence tests.

$$\Rightarrow \quad f \in H\big(D(z_1, R_1) \cup D(z_2, R_2) \cup \dots\big).$$

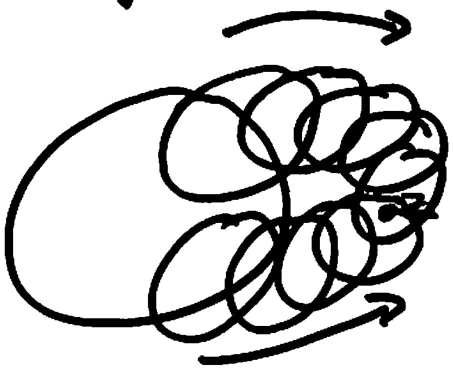
- <u>Warning</u>: AC not automatically possible and in general an ill-posed problem.

- E.g. $f(z) = \sum_{n=0}^{\infty} z^{n!} \in H(D(0,1))$, but has

  a dense set of singularities on the unit circle (a "natural barrier") at $z = e^{i\theta}$, $\theta = 2\pi p/q$ for $p, q \in \mathbb{Z}$ ($\because e^{i\theta n!} = 1 \ \forall n \geq q$).

- <u>Identity Theorem</u>: A holomorphic function on a region D is completely determined by its value on any set of points $S \subseteq D$ containing an accumulation or limit point.

$\Rightarrow$ AC via the circle-chain method is locally unique - for global uniqueness need the <u>Monodromy Theorem</u>:



Same answer at z iff f is holomorphic between the chains.

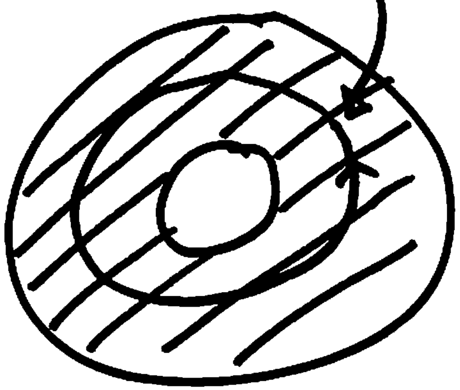
- <u>Warning</u>: if f is multivalued (e.g. $z^{1/2}$, $\log z$), then the circle-chain method generates all branches.

- <u>Isolated zeros</u>: if the zeros of a non-constant function $f \in H(D)$ have an accumulation point, then $f \equiv 0$ by the Identity Theorem, hence zeros must be isolated.

- **Laurent's Theorem**: $f \in H(\{z \in \mathbb{C} : 0 \le \varsigma < |z-a| < R \le \infty\})$

$$\Rightarrow \quad f(z) = \underbrace{\sum_{n=-\infty}^{-1} c_n (z-a)^n}_{\substack{\text{Principal part} \\ \in H(\{z \in \mathbb{C} : |z-a| > \varsigma\})}} + \underbrace{\sum_{n=0}^{\infty} c_n (z-a)^n}_{\substack{\text{Fundamental part} \\ \in H(\{z \in \mathbb{C} : |z-a| < R\})}}$$

where $$c_n = \frac{1}{2\pi i} \int_{|z-a|=\rho} \frac{f(t)}{(t-a)^{n+1}} \, dt \quad (\varsigma < \rho < R)$$



Classification of singularities

- $\delta = 0 \Rightarrow f(z)$ has an isolated singularity at $z=a$, which is

Ⓐ <u>remarable</u> if $c_n = 0$ $\forall_{n<0}$ (by defining $f(a) = c_0$);

Ⓑ a <u>pole of order m</u> if $c_{-m} \neq 0$ but $c_n = 0$ $\forall_{n<-m<0}$;

Ⓒ an <u>essential singularity</u> if $\nexists -m<0$ s.t. $c_n = 0$ $\forall_{n<-m}$

- NB: Use $z \mapsto 1/z$ to classify singularities at ∞.
  e.g. Ⓑ for z^m, Ⓒ for e^z.

- $c_{-1} := \operatorname{res}_a f(z)$, the <u>residue</u> of $f(z)$ at $z=a$.

- <u>Cauchy's Residue Theorem</u>: $f \in H(\Gamma \cup \mathrm{Int}(\Gamma) \setminus \{a_j\}_{1 \le j \le N})$

$$\Rightarrow \oint_\Gamma f(z)\,dz = 2\pi i \sum_{j=1}^{N} \operatorname{res}_{a_j} f(z)$$

- Calculate residues using Laurent or Taylor series expansions, e.g.

$$f(z) = \frac{g(z)}{(z-a)^{n+1}}, \quad g \in H(D(a,\varepsilon)), \quad n \in \mathbb{N}$$

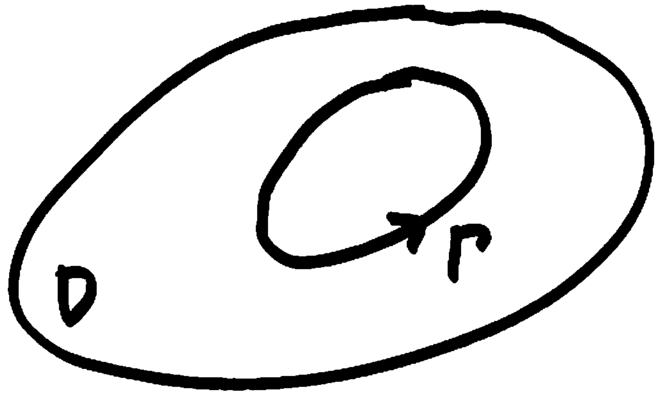
$$\Rightarrow \operatorname{res}_a f(z) = \frac{g^{(n)}(a)}{n!}.$$

Partial converse to Cauchy's Theorem:

Morera's Theorem

$f \in C(D)$ and $\oint_\Gamma f(z)\,dz = 0$

$\forall$ closed $\Gamma \subseteq D \Rightarrow f \in H(D)$



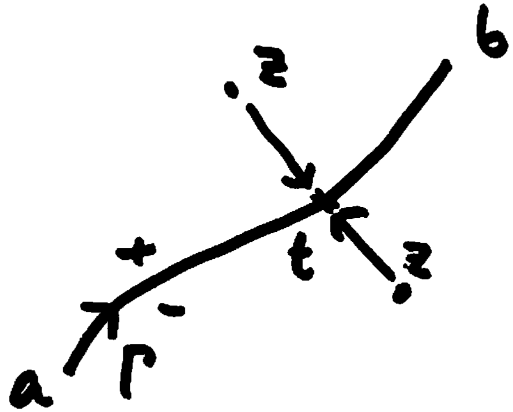
Corollary

If (I) Γ is open and does not contain its end points a, b;

(II) $w \in H(\mathbb{C} \setminus \Gamma \cup \{a, b\})$;

(III) $w_\pm(t) = \lim_{z \to t} w(z)$ from $\pm$ side of Γ,

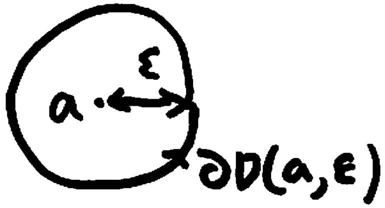
s.t. $w_+ = w_-$ on Γ and $w_\pm \in C(\Gamma)$.



Then, $w \in H(\mathbb{C} \setminus \{a, b\})$.

Multifunctions

- $f(z)$ has a branch point at $z = a$ iff $\exists\, \varepsilon_0 > 0$ s.t. $f \notin C(\partial D(a,\varepsilon))$ for all $0 < \varepsilon < \varepsilon_0$



- $z = \infty$ a branch point iff $z = 0$ is a branch point of $f(1/z)$.

- Join up branch points with curves or branch cuts to restrict the domain of definition and select thereby a single-valued and continuous branch of a multifunction.

- cf. Riemann surface approach, which extends the domain of definition by "stacking" $\mathbb{C}$-planes.

Example: $w = \log z$

$\Rightarrow e^w = z$, so let $\quad w = u + iv \quad (u, v \in \mathbb{R})$

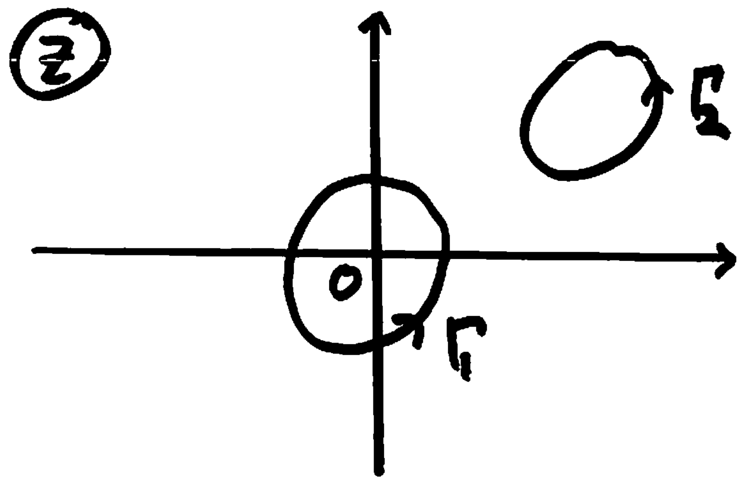
$\qquad\qquad\qquad\qquad\quad z = re^{i\theta} \quad (r > 0, \theta \in \mathbb{R})$

$\Rightarrow e^u e^{iv} = re^{i\theta}$

$\Rightarrow u = \log r, \; v = \theta + 2k\pi \quad (k \in \mathbb{Z})$

$\Rightarrow w = w_k := \log r + i(\theta + 2k\pi) \quad (k \in \mathbb{Z})$

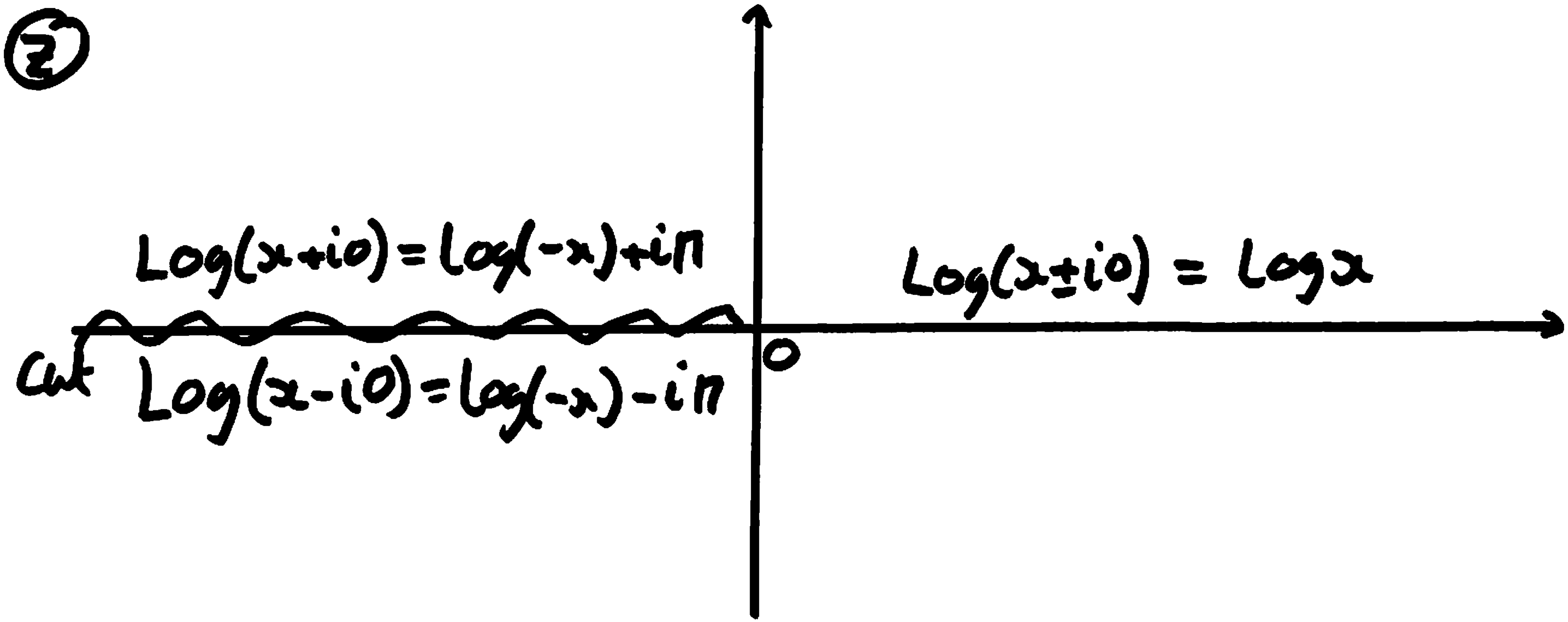
$\Rightarrow$ infinite number of branches.



$$0 \notin \mathrm{Int}(\Gamma_2) \Rightarrow [\theta]_{\Gamma_2} = 0$$
$$\Rightarrow w_k \in C(\Gamma_2)$$
$$0 \in \mathrm{Int}(\Gamma_1) \Rightarrow [\theta]_{\Gamma_1} = 2\pi$$
$$\Rightarrow w_k \notin C(\Gamma_1)$$

- Hence, $z = 0$ is a branch point of $\log z$.
- $e^{-w} = 1/z \Rightarrow \log(1/z) = -\log z \Rightarrow z = \infty$ is a branch point of $\log z$.
- To select a branch: choose (i) $k \in \mathbb{Z}$ and (ii) cut plane from $z = 0$ to $z = \infty$, e.g. by restricting the domain of θ.

- Principal branch $\mathrm{Log}\, z := \log r + i\theta \quad (r>0,\ -\pi<\theta\le\pi)$



$$\Rightarrow \quad \mathrm{Log} \in H(\mathbb{C}\setminus(-\infty,0]), \quad \frac{d}{dz}\mathrm{Log}\, z = \frac{1}{z}.$$

Example: $w = z^{1/n}$ $(n \in \mathbb{N})$

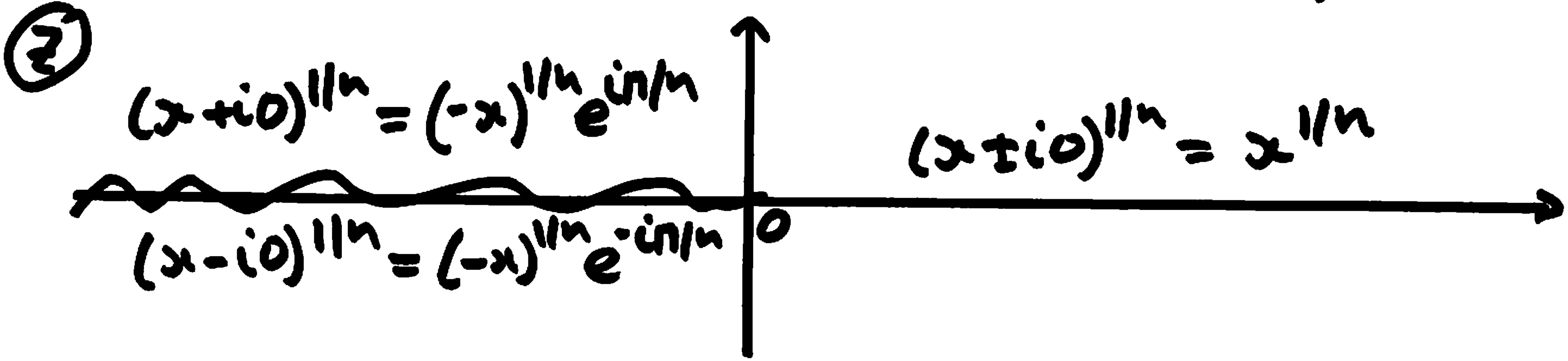
- $z^{1/n} := \exp\left(\frac{1}{n}\log z\right)$

$\Rightarrow$ Branch points at $z = 0$ and $z = \infty$.

- $\log z = w_k \Rightarrow z^{1/n} = \exp\left(\frac{1}{n} w_k\right) = \exp\left(\frac{2k\pi i}{n}\right) r^{1/n} e^{i\theta/n}$

$\Rightarrow$ n distinct branches (e.g. $k = 0, 1, \ldots, n-1$).

- Principal branch: $z^{1/n} := \exp\left(\frac{1}{n}\operatorname{Log} z\right)$

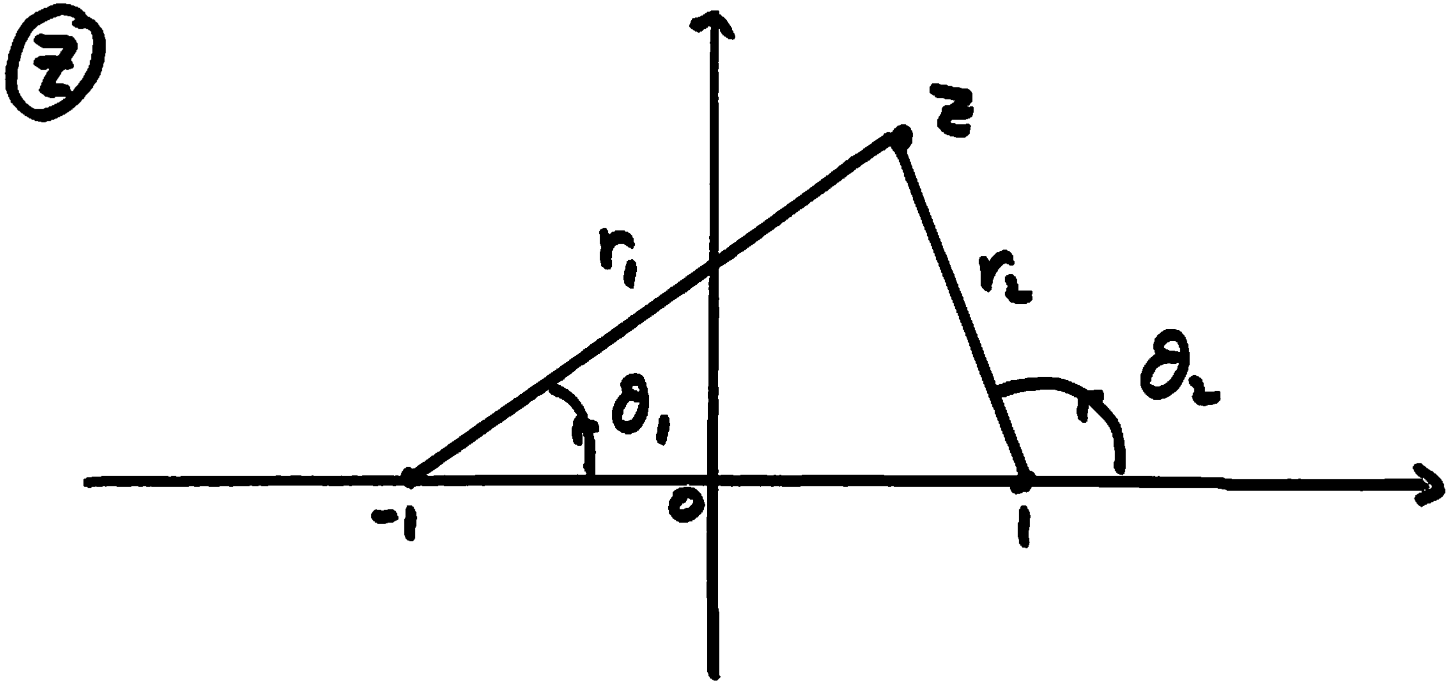


Example: $w = (z^2-1)^{1/2}$

$\Rightarrow \quad w^2 = (z+1)(z-1)$, so let $w = Re^{i\Theta}$ $(R>0, \Theta \in \mathbb{R})$

$$z+1 = r_1 e^{i\theta_1} \quad (r_1>0, \theta_1 \in \mathbb{R})$$

$$z-1 = r_2 e^{i\theta_2} \quad (r_2>0, \theta_2 \in \mathbb{R})$$



$$\Rightarrow \quad R^2 e^{2i\Theta} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

$$\Rightarrow \quad R = (r_1 r_2)^{1/2}, \quad \Theta = \tfrac{1}{2}(\theta_1+\theta_2) + k\pi \quad (k \in \mathbb{Z})$$

$$\Rightarrow \quad w = w_k := (-1)^k (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2} \quad (k \in \mathbb{Z})$$

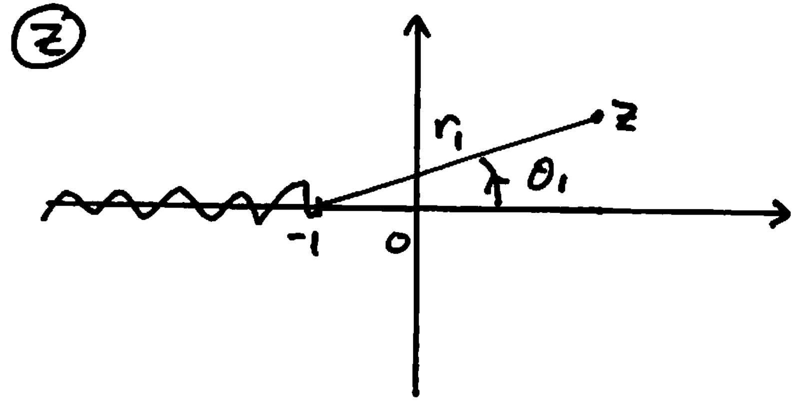
$\Rightarrow$ Two branches (k odd or even $\Rightarrow \pm$) and branch points at $z = \pm 1$, but not $z = \infty$.

- To select a branch: choose (i) $\pm$ and (ii) cut plane from $z=-1$ to $z=+1$ (through $z=\infty$ OK).

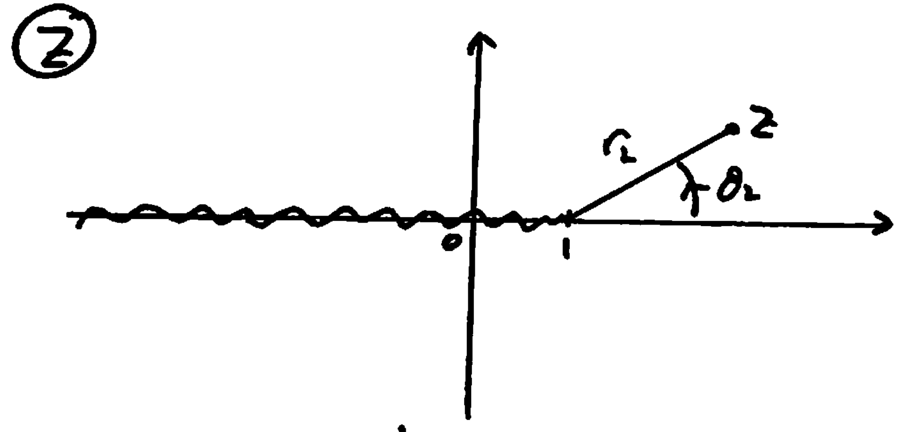
- Since $(z^2-1)^{1/2} = (z-1)^{1/2}(z+1)^{1/2}$ this is equivalent to choosing a branch for each of $(z-1)^{1/2}$ and $(z+1)^{1/2}$.

- In this course, there are two useful cases.

Case ⓐ



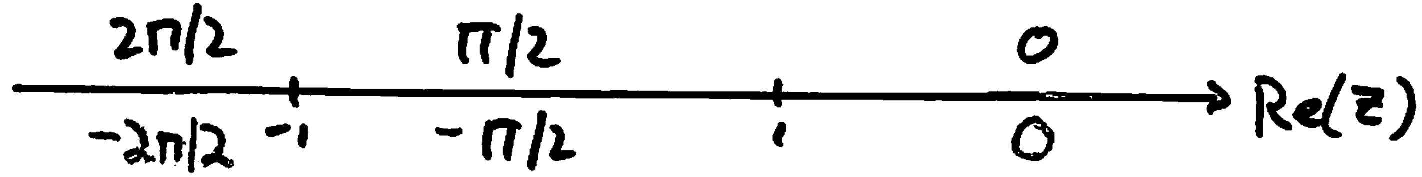
$(z+1)^{1/2} := +r_1^{1/2} e^{i\theta_1/2} \quad (r_1 > 0, -\pi < \theta_1 \le \pi)$



$(z-1)^{1/2} := +r_2^{1/2} e^{i\theta_2/2} \quad (r_2 > 0, -\pi < \theta_2 \le \pi)$

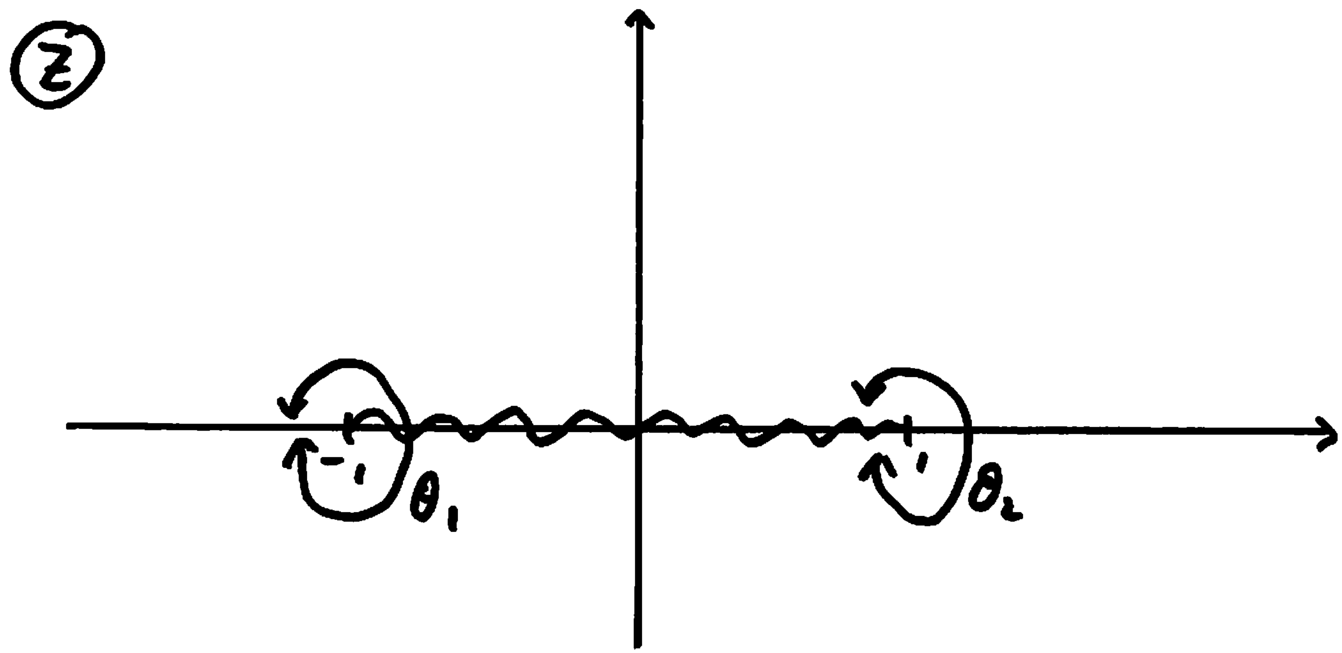
$$\Rightarrow \quad (z^2-1)^{1/2} := +(r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2} \quad (r_1, r_2 > 0,\ -\pi < \theta_1, \theta_2 \le \pi)$$

$$\Rightarrow \quad \frac{\theta_1+\theta_2}{2} :$$



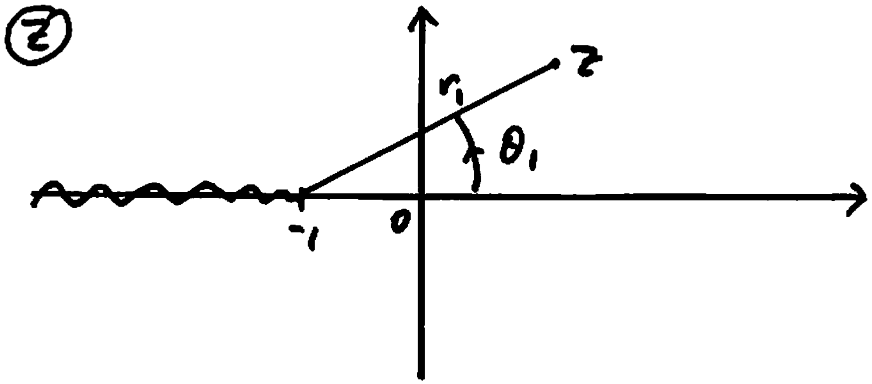
$$\Rightarrow \quad (z^2-1)^{1/2} = \begin{cases} (x^2-1)^{1/2} & \text{for } z = x \pm i0,\ x>1 \\ \pm i(1-x^2)^{1/2} & \text{for } z = x \pm i0,\ |x|<1 \\ -(x^2-1)^{1/2} & \text{for } z = x \pm i0,\ x<-1 \end{cases}$$

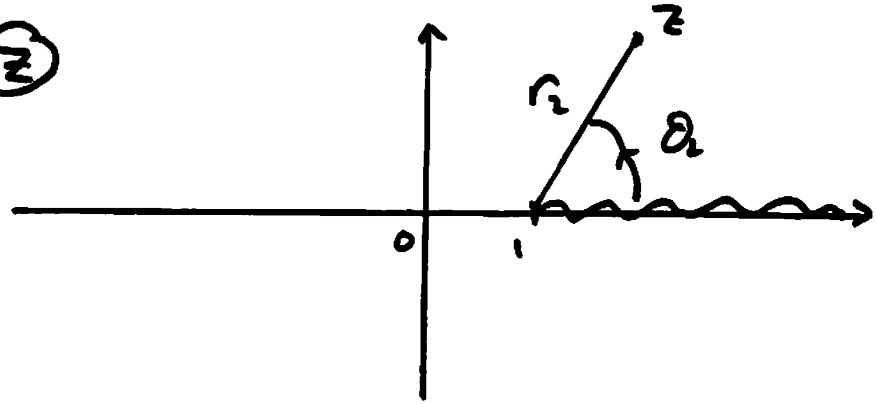
$\Rightarrow$



NB: Contours cannot cross cut, but θ_1, θ_2 can.

Case (b)

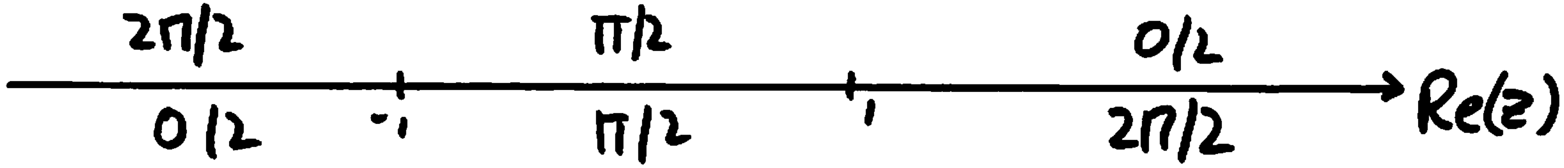




$(z+1)^{1/2} := +r_1^{1/2} e^{i\theta_1/2} \quad (r_1 > 0, -\pi < \theta_1 \leq \pi)$ $\qquad$ $(z-1)^{1/2} := +r_2 e^{i\theta_2/2} \quad (r_2 > 0, 0 < \theta_2 \leq 2\pi)$

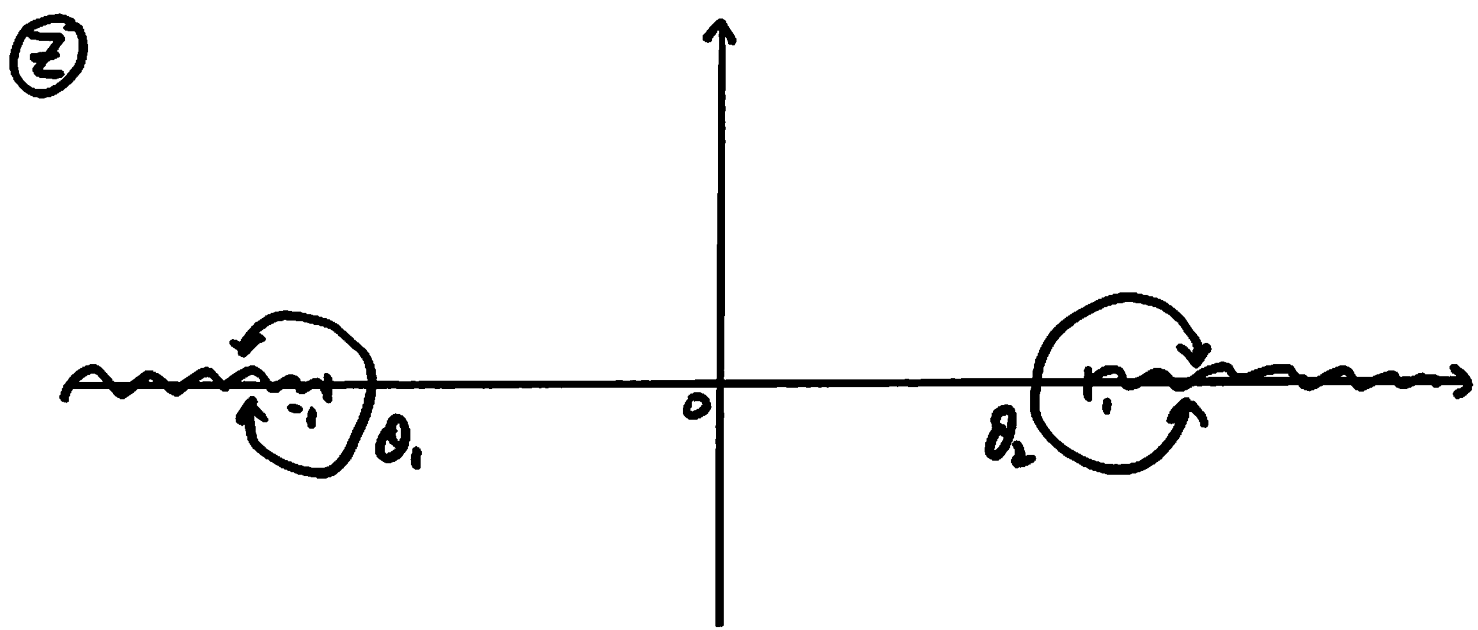
$$\Rightarrow (z^2-1)^{1/2} := +(r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2)/2} \quad (r_1, r_2 > 0, -\pi < \theta_1 \leq \pi, 0 < \theta_2 \leq 2\pi)$$

$$\Rightarrow \frac{\theta_1+\theta_2}{2} :$$



$$\Rightarrow \quad (z^2-1)^{1/2} = \begin{cases} \pm(x^2-1)^{1/2} & \text{for } z = x \pm i0,\ x>1 \\ i(1-x^2)^{1/2} & \text{for } z = x \pm i0,\ |x|<1 \\ \mp(x^2-1)^{1/2} & \text{for } z = x \pm i0,\ x<-1 \end{cases}$$

$\Rightarrow$



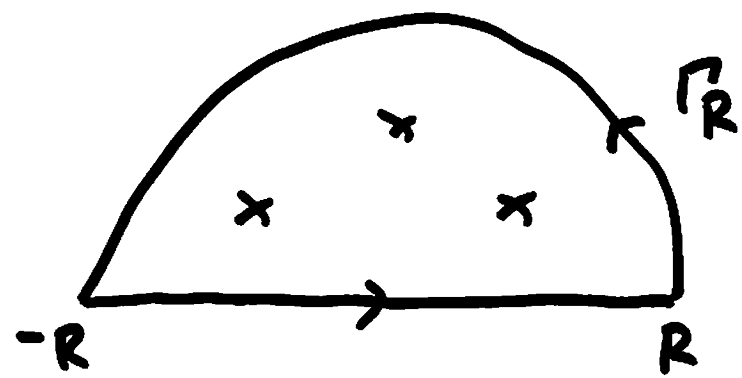
NB: OK for cut to pass through $z = \infty$ even if $z = \infty$ is not a branch point.

Evaluation of integrals

① $\displaystyle\int_0^{2\pi} F(\sin\theta)\,d\theta \mapsto \int_{\partial D(0,1)} F\Big(\underbrace{\frac{z-1/z}{2i}}_{z=e^{i\theta}}\Big)\,dz$

② $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx \quad (\deg(Q) \geq \deg(P)+2,\ Q \neq 0 \text{ on } \mathbb{R})$

$\displaystyle\mapsto \int_{\Gamma_R} \frac{P(z)}{Q(z)}\,dz$ as $R \to \infty$



③ $\displaystyle\int_{-\infty}^{\infty} \frac{P(x)\cos x}{Q(x)}\,dx \longmapsto \int_{\Gamma_R} \frac{P(z)e^{iz}}{Q(z)}\,dz$ on same contour

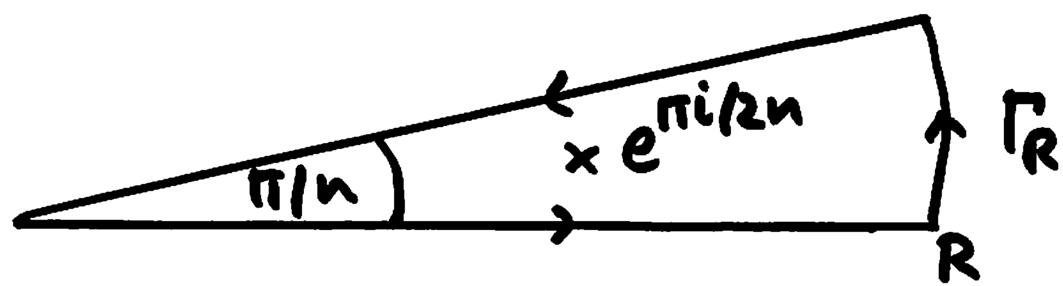
NB: Close at $z = +i\infty$ $\because$ $e^{iz} = e^{ix-y} \to 0$ as $y \to \infty$

NB: Integral may exist if $\deg(P) = \deg(Q) - 1$,
but need to use Jordan's inequality,

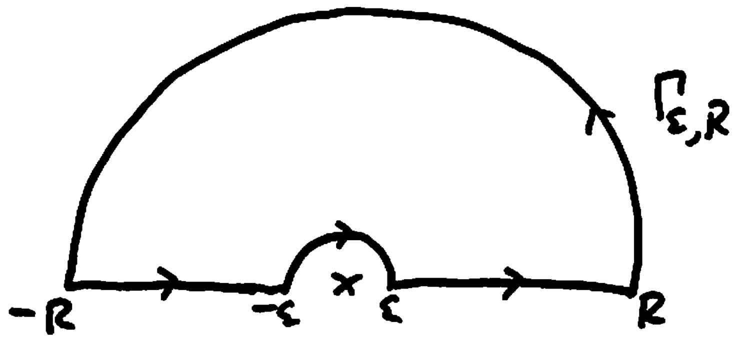
$$\frac{2}{\pi} \leq \frac{\sin\theta}{\theta} \leq 1 \quad \text{for } 0 \leq \theta \leq \frac{\pi}{2},$$

to estimate contribution from large semi-circle

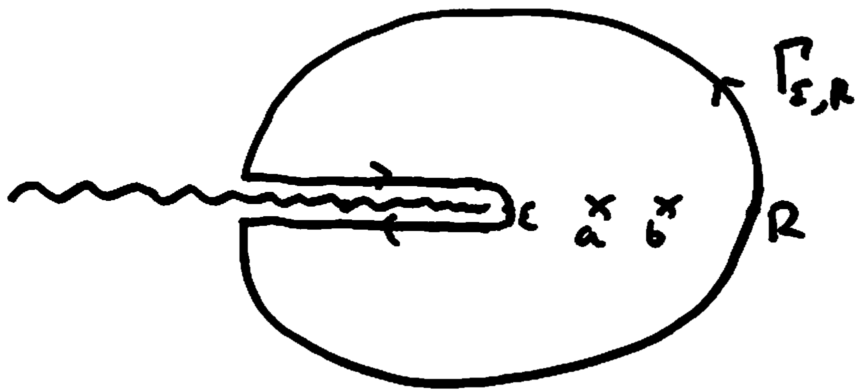
④ $\displaystyle\int_0^{\infty} \frac{dx}{1+x^{2n}}$ $(n \in \mathbb{N}) \mapsto \displaystyle\int_{\Gamma_R} \frac{dz}{1+z^{2n}}$ as $R \to \infty$



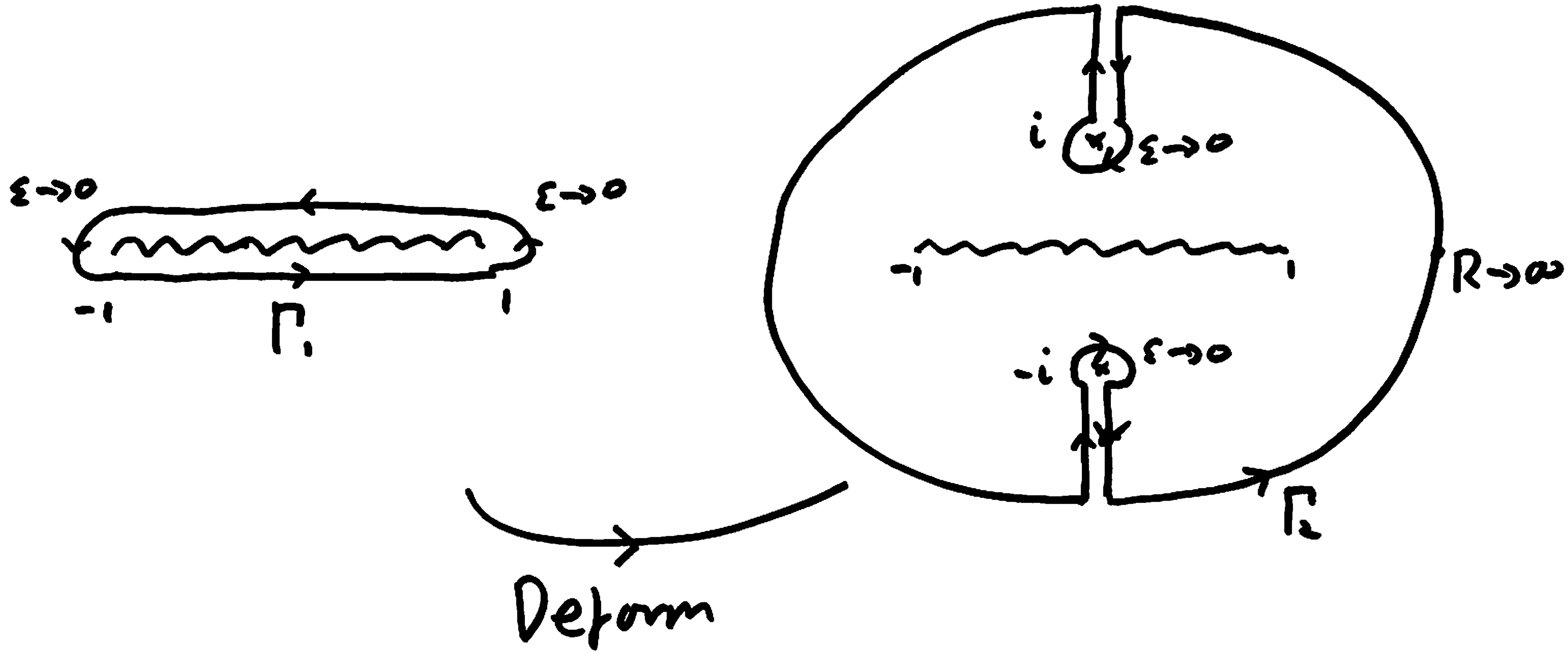
⑤ $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx \longmapsto \int_{\Gamma_{\varepsilon,R}} \frac{e^{iz}}{z}\,dz$ as $\varepsilon \to 0$, $R \to \infty$



⑥ $\displaystyle\int_0^\infty \frac{\log x}{(x+a)(x+b)}\,dx \quad (a,b>0) \longmapsto \int_{\Gamma_{\varepsilon,R}} \frac{(\mathrm{Log}\, z)^2}{(z-a)(z-b)}\,dz$ as $\varepsilon \to 0$, $R \to \infty$



⑦ $$\int_{-1}^{1} \frac{\sqrt{1-x^2}}{1+x^2}\,dx \longmapsto \int_{\Gamma_j} \frac{\sqrt{1-z^2}}{1+z^2}\,dz \quad (j=1,2)$$



Fourier Transforms

Convergence Theorem for Fourier Series

If $f : \mathbb{R} \to \mathbb{C}$ is piecewise smooth (i.e. f, f' are piecewise continuous on all closed bounded subintervals of $\mathbb{R}$) and periodic with period $2L$, then

$$\frac{1}{2}\left(f(x-) + f(x+)\right) = \sum_{n=-\infty}^{\infty} c_n e^{-in\pi x/L}$$

where $f(x-) = \lim_{\tilde{x} \uparrow x} f(\tilde{x})$, $f(x+) = \lim_{\tilde{x} \downarrow x} f(\tilde{x})$ (so $f(x\pm) = f(x)$ iff f is continuous at x) and

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{in\pi x/L} \, dx.$$

- Remove requirement that f be periodic by letting $L \to \infty$ with $k = \frac{n\pi}{L} = O(1)$

- If $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$, then

$$2Lc_n = \int_{-L}^{L} f(x) e^{ikx}\, dx \to \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx \quad \text{as } L \to \infty.$$

- This is the Fourier transform of f:

$$\bar{f}(k) \equiv F(f) := \int_{-\infty}^{\infty} f(x) e^{ikx}\, dx.$$

- To invert note that

$$\frac{1}{2}\left(f(x-) + f(x+)\right) = \lim_{L \to \infty} \sum_{n=-\infty}^{\infty} \frac{1}{2L} \bar{f}\left(\frac{n\pi}{L}\right) e^{-in\pi x/L}$$

$$= \lim_{h \to 0} \frac{h}{2\pi} \sum_{n=-\infty}^{\infty} \bar{f}(nh) e^{-inhx}, \quad \text{where } h = \frac{\pi}{L}.$$

- It may be shown that this leads to...

Fourier Inversion Theorem (FIT)

$f : \mathbb{R} \to \mathbb{C}$ piecewise smooth, $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

$$\Rightarrow \quad \frac{1}{2}\left(f(x-) + f(x+)\right) = \frac{1}{2\pi} \fint_{-\infty}^{\infty} \tilde{f}(k)\, e^{-ikx}\, dk,$$

where the principal value integral

$$\fint_{-\infty}^{\infty} := \lim_{R \to \infty} \int_{-R}^{R}.$$

Key properties

① Linearity $\quad F[\mu f + \lambda g] = \mu F[f] + \lambda F[g]$

② $F[f'] = -ik\bar{f}$.

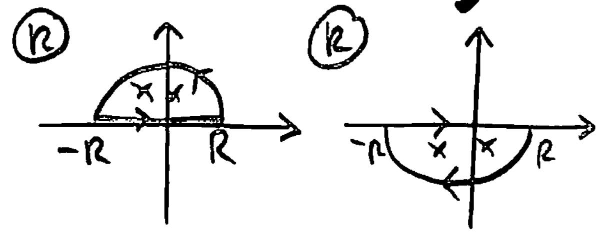
③ $F[xf(x)] = -i\dfrac{d\bar{f}}{dk}$.

④ $f*g(x) = \int_{-\infty}^{\infty} f(x-s)g(s)\,ds \Rightarrow F[f*g] = \bar{f}\,\bar{g}$.

⑤ In applications, $\bar{f}(k)$ given by a formula

$\Rightarrow$ can often AC into complex k-plane, except for singularities or branch cuts.

$\Rightarrow$ can often use contour integration (i.e. CRT) to invert, closing the contour at $k = \pm i\infty$ only if $\bar{f}(k)e^{-ikx} \to 0$ as $k \to \pm i\infty$.



Laplace Transforms

- If $f : \mathbb{R}^+ \to \mathbb{C}$ is piecewise smooth and $\exists\, c \in \mathbb{R}$ s.t. $|f(x)| = O(e^{cx})$ as $x \to \infty$, then the Laplace transform of f is defined by

$$\hat{f}(p) \equiv L[f] := \int_0^\infty f(x) e^{-px}\, dx \quad \text{for } \mathrm{Re}(p) > c.$$

- The inversion formula is

$$\frac{1}{2}\left(f(x-) + f(x+)\right) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \hat{f}(p) e^{px}\, dx,$$

where $\gamma > c$, so that the inversion contour is to the right of any singularities of $\hat{f}(p)$.

Key properties

① Linearity.

② $L[f'] = p\hat{f}(p) - f(0)$ for $\mathrm{Re}(p) > c$.

③ $\hat{f} \in H(\{p \in \mathbb{C} : \mathrm{Re}(p) > c\})$, with $\frac{d\hat{f}}{dp} = L[-x f(x)]$.

④ $f * g(x) := \int_0^x f(x-s) g(s)\, ds \Rightarrow L[f*g] = \hat{f}\,\hat{g}$.

⑤ In applications, $\hat{f}(p)$ given by a formula $\Rightarrow$ use AC and contour integration to invert.

⑥ We'll see Laplace transforms are just a special case of Fourier transforms if you allow complex k.

Conformal mapping

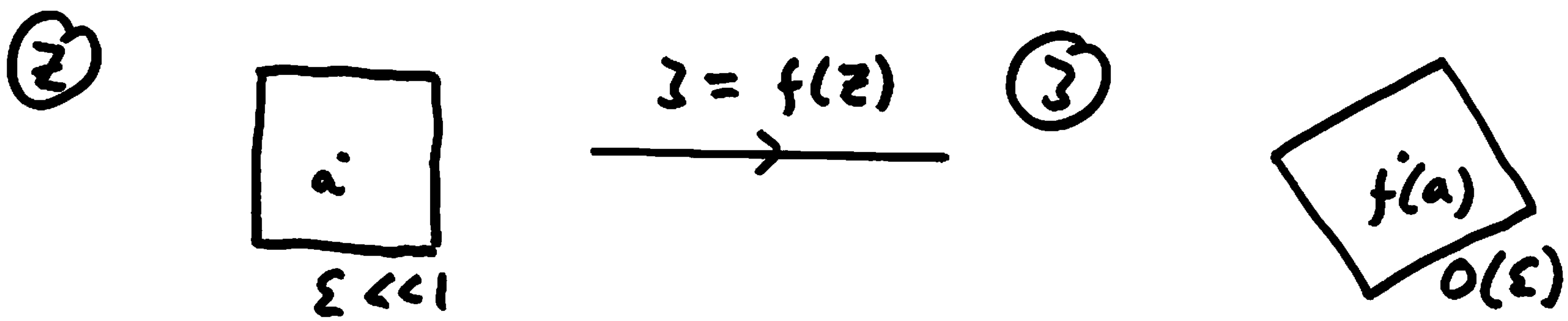
- $f \in H(D)$ and $f'(z) \neq 0 \ \forall z \in D \Rightarrow f$ is a conformal map, i.e. it preserves angles.

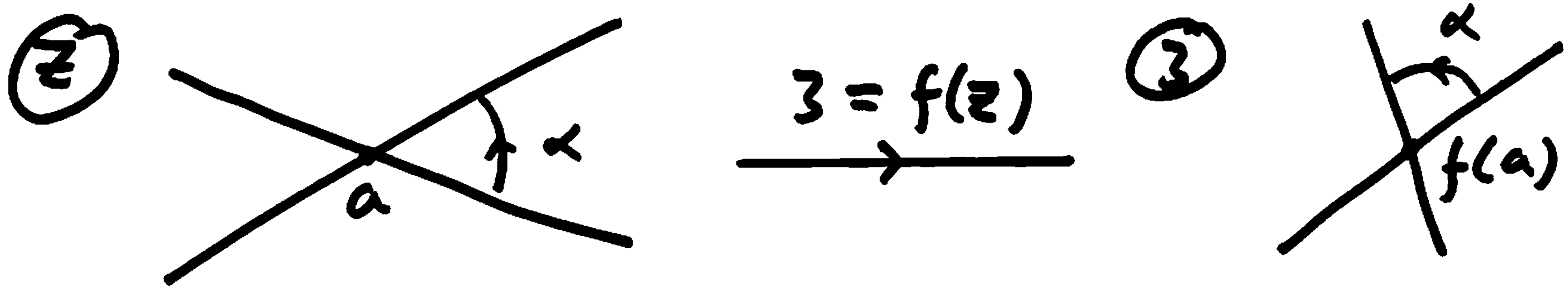
- To see this, use Taylor's Theorem : $a \in D$

$$\Rightarrow \zeta = f(z) = \underbrace{f(a)}_{\text{translation}} + \underbrace{f'(a)(z-a)}_{\text{rotation \& scaling}} + \cdots \quad \text{as } z \to a.$$

$\Rightarrow$ Locally, map is linear and one-to-one.

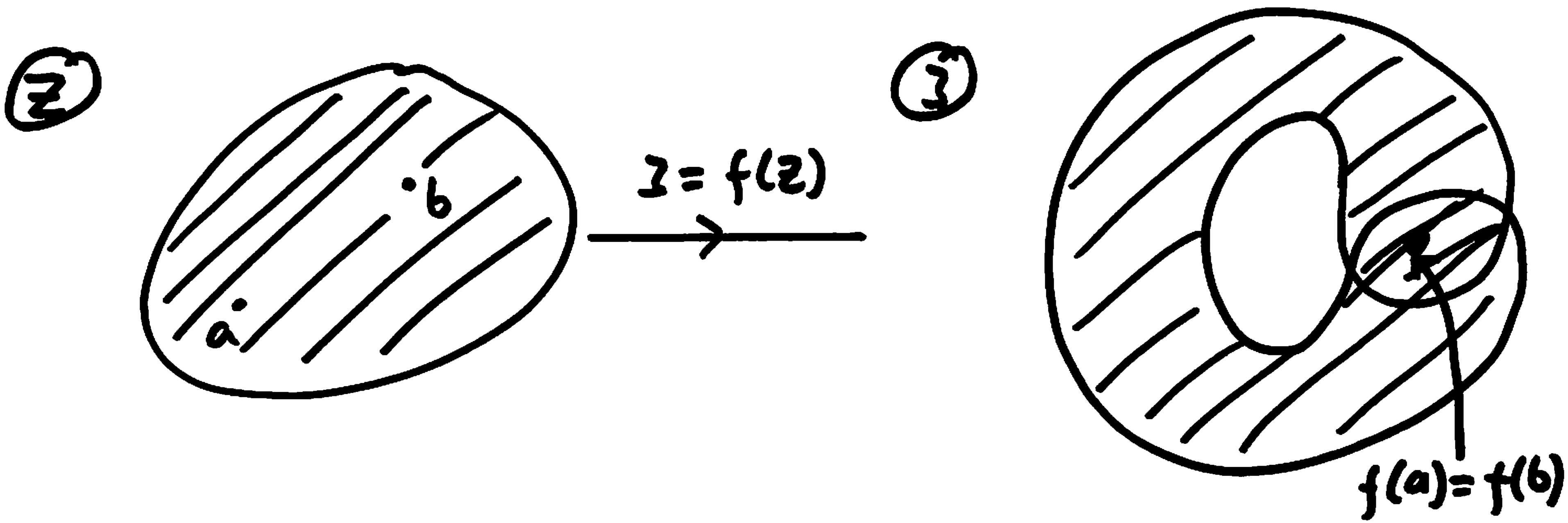
$\Rightarrow$





Remarks

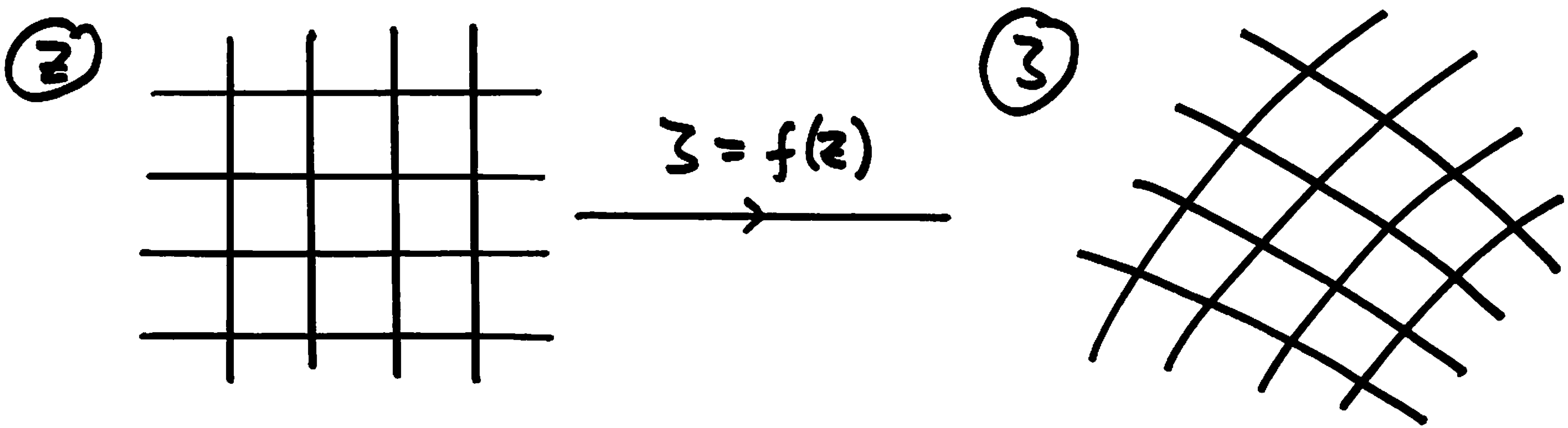
① Not necessarily one-to-one globally, e.g.



(2) Composition of conformal maps is conformal

$\Rightarrow$ need familiarity with standard maps to use as building blocks.

(3) All orthogonal curvilinear coordinates can be generated by a conformal map of Cartesian coordinates.



Behaviour near a critical point

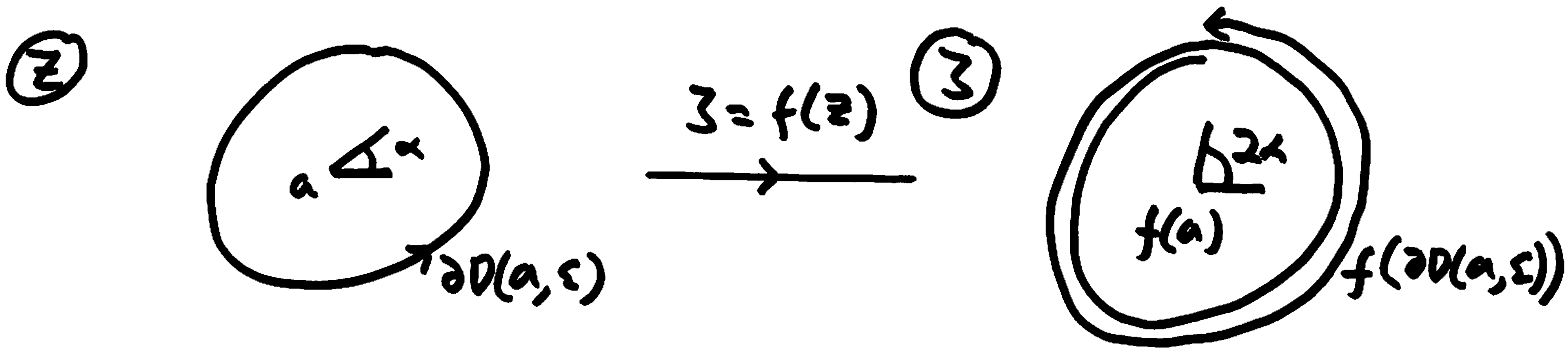
- $a \in D \cup \partial D$ is a <u>critical point</u> iff $f'(a) = 0$.

- If $f'(a) = 0 \neq f''(a)$, then

$$f(z) - f(a) = \frac{1}{2} f''(a)(z-a)^2 + O((z-a)^3) \text{ as } z \to a.$$

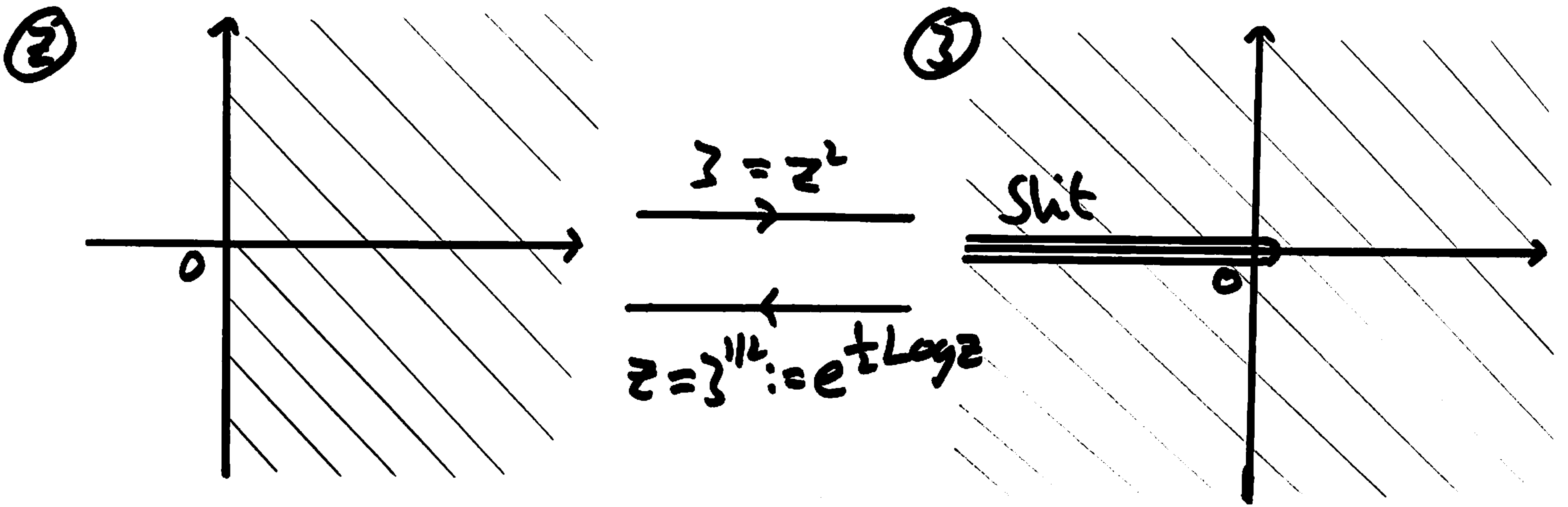
- If $z - a = \varepsilon e^{i\theta}$, then $\zeta - f(a) = \frac{1}{2} f''(a) \varepsilon^2 e^{2i\theta} + O(\varepsilon^3)$ as $\varepsilon \to 0$

$\Rightarrow$ angles doubled and map not one-to-one locally.



- Hence, need $a \in \partial D$ if map $f : D \to \tilde{D} \subseteq \mathbb{C}$ is to be one-to-one.

- E.g. $\zeta = f(z) = z^2$ has a critical point at $z = 0$.



$$D = \{z \in \mathbb{C} : |\arg z| < \tfrac{\pi}{2}\} \quad \longleftrightarrow \quad \tilde{D} = \{\zeta \in \mathbb{C} : |\arg \zeta| < \pi\}$$

one-to-one

- In general, local behaviour of $f(\partial D)$ near a critical point depends delicately on the curvature of ∂D near $z = a$ and on the derivatives of $f(z)$ at $z = a$.

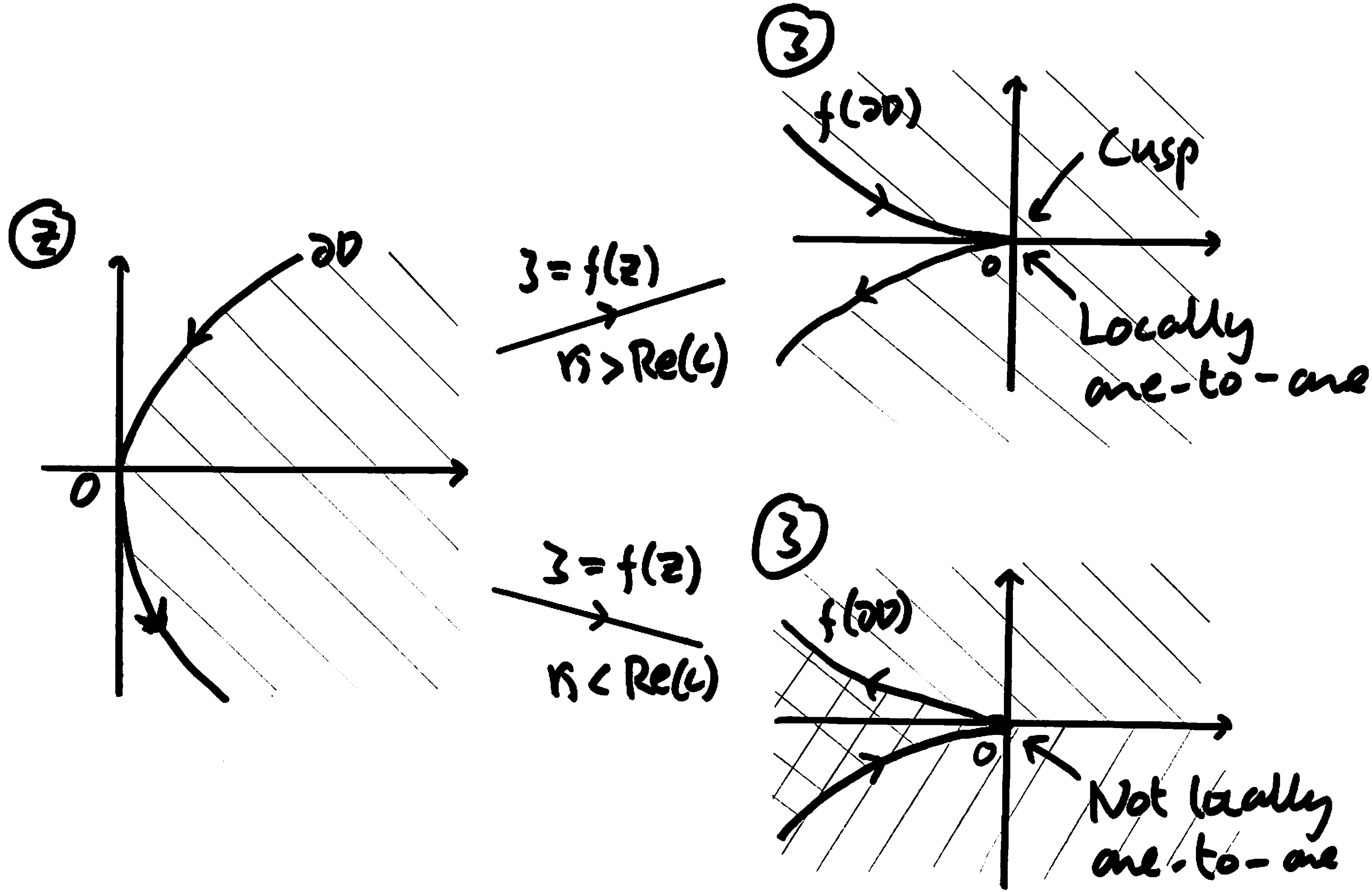
- E.g. Suppose (i) ∂D: $z \sim it + \frac{1}{2}\kappa t^2$ as $t \to 0$ $(\kappa \in \mathbb{R})$;

  (ii) $f(z) \sim z^2 + cz^3$ as $z \to 0$ $(c \in \mathbb{C})$.

  Then on $f(\partial D)$ near $f(0)$,

$$\zeta = \xi + i\eta = f(z) \sim \left(it + \tfrac{1}{2}\kappa t^2\right)^2 + c\left(it + \tfrac{1}{2}\kappa t^2\right)^3 \text{ as } t \to 0$$

$$\Rightarrow \quad \xi = -t^2 + O(t^3), \quad \eta = (\kappa - \mathrm{Re}(c))t^3 + O(t^4) \text{ as } t \to 0.$$


z
∂D
0
ζ = f(z)
κ > Re(c)
ζ = f(z)
κ < Re(c)
ζ
f(∂D)
Cusp
0
Locally one-to-one
ζ
f(∂D)
0
Not locally one-to-one

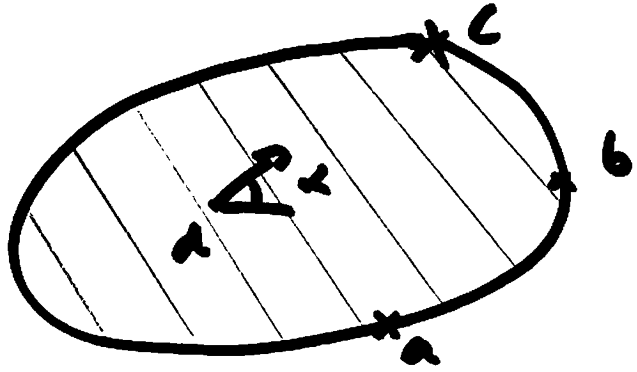
Riemann Mapping Theorem

If $D \neq \mathbb{C}$ is a simply connected region, then there exists a three-parameter family of conformal transformations $f : D \to D(0,1)$.

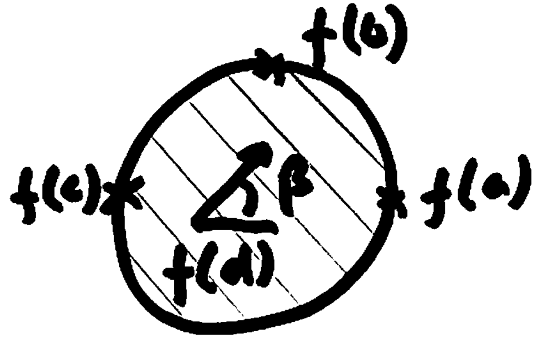
Remarks

1. If ∂D is a contour, obtain a unique transformation by prescribing e.g. 3 ordered boundary points on $f(\partial D)$ or an interior point and a direction at that point.

2.



3.



② 3-parameter family ∵ ∃ a 3-parameter family of maps from $D(0,1) \to D(0,1)$, given by

$$\zeta = e^{i\phi} \frac{z-\omega}{1-\bar{\omega} z} \qquad (\omega \in \mathbb{C}, \phi \in \mathbb{R})$$

③ $D = \mathbb{C} \Rightarrow f \in H(\mathbb{C})$ and $|f| \leq 1$ on $\mathbb{C}$

$\Rightarrow f =$ constant by Liouville $\lightning$

④ Often use upper-half plane as $f(D)$, with e.g. $f(a) = 0$, $f(b) = 1$, $f(c) = \infty$.

⑤ f may be crazy on ∂D ∵ singularities are required to smooth any corners and cusps.

⑥ Orientation preserved (∵ conformal maps preserve angles): very useful in practice!

Standard maps

Bilinear maps (Möbius transformations)

- $\zeta = f(z) = \dfrac{az+b}{cz+d}$, where $a, b, c, d \in \mathbb{C}$, $ad - bc \neq 0$, $c \neq 0$

  ($ad-bc \neq 0$: $f \neq$ constant; $c \neq 0$: nontrivial.)

- $f \in H(\mathbb{C} \setminus \{-d/c\})$ with $f'(z) = \dfrac{ad-bc}{(cz+d)^2} \neq 0 \quad \forall z \neq -d/c$.

- $f : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ is one-to-one upon defining $f(\infty) = \frac{a}{c}$, $f(-\frac{d}{c}) = \infty$.

- Composition of

$$\zeta_1 = cz + d, \qquad \zeta_2 = 1/\zeta_1, \qquad \zeta = \frac{a}{c} + \frac{bc-ad}{c}\zeta_2$$

$\zeta_1 = cz+d$: Rotation, scaling and translation

$\zeta_2 = 1/\zeta_1$: Inversion

$\zeta = \frac{a}{c} + \frac{bc-ad}{c}\zeta_2$: Rotation, scaling and translation

- Map circlines to circlines.

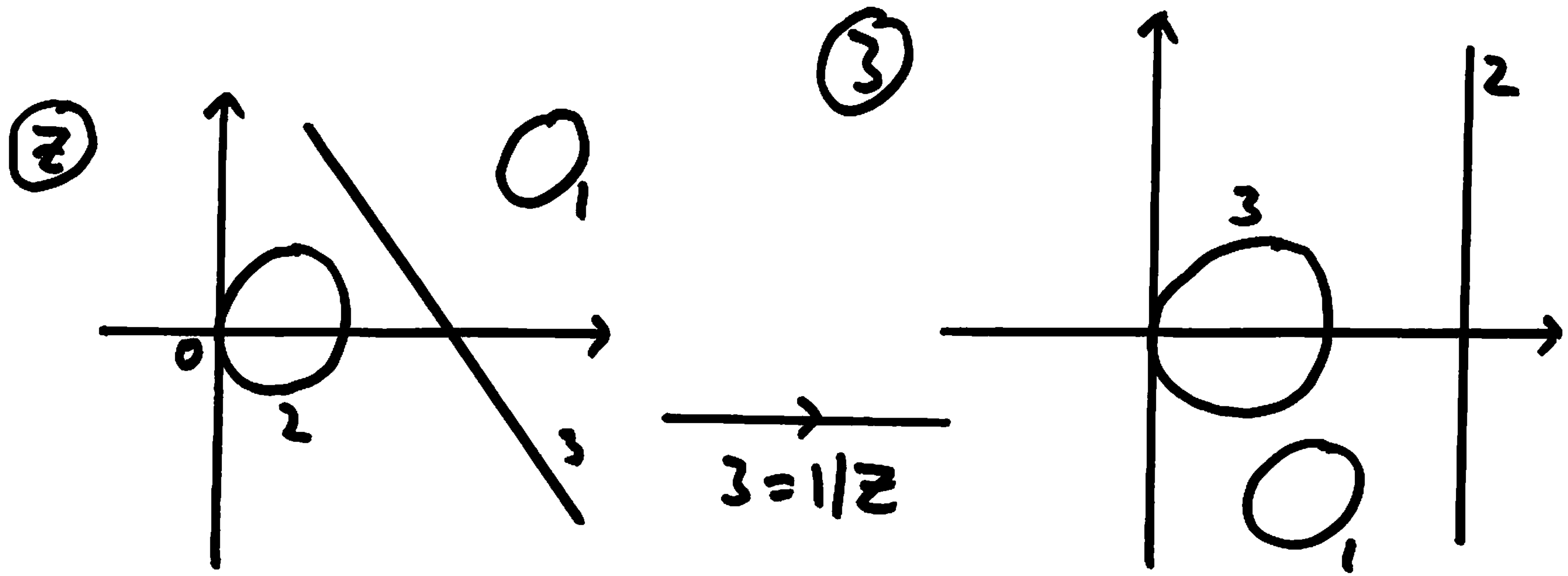
Pf: Trivial for rotations, scalings and translations, so it is sufficient to show for inversions:

$$\alpha^2 z\bar{z} + \bar{\beta} z + \beta \bar{z} + \gamma^2 = 0 \quad (\alpha, \gamma \in \mathbb{R}, \beta \in \mathbb{C};\ \text{circle } \alpha \neq 0,\ \text{line } \alpha = 0)$$

$$\Rightarrow \alpha^2 + \bar{\beta}\bar{\zeta} + \beta\zeta + \gamma^2 \zeta\bar{\zeta} = 0, \text{ i.e. } \alpha \leftrightarrow \gamma,\ \beta \to \bar{\beta} \qquad \square$$

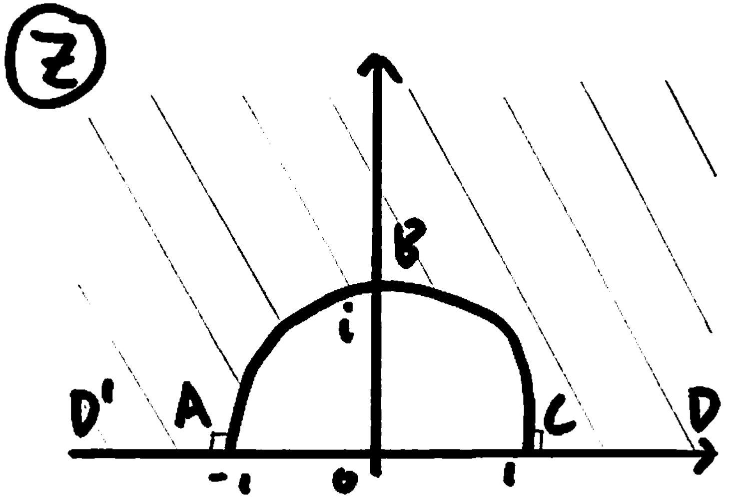
$\zeta = 1/z$

- Eg



- Thus, if $\partial D = \bigcup\{\text{circlines}\}$, then $f(\partial D) = \bigcup\{\text{circlines}\}$, with the same angles and orientation at corners where the circlines meet.

- E.g.

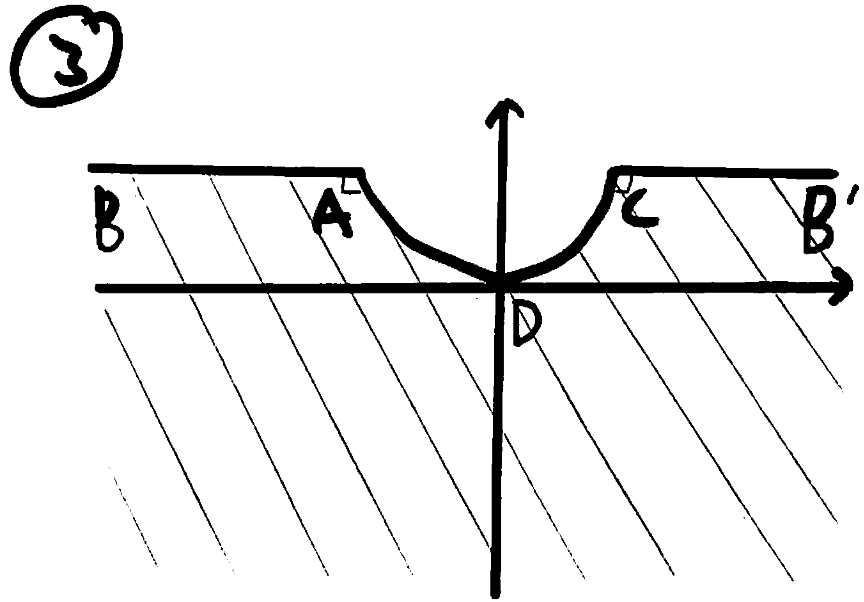


$$\zeta = \frac{1}{z-i}$$

$$\pm 1 \to \frac{\pm 1 + i}{2}$$

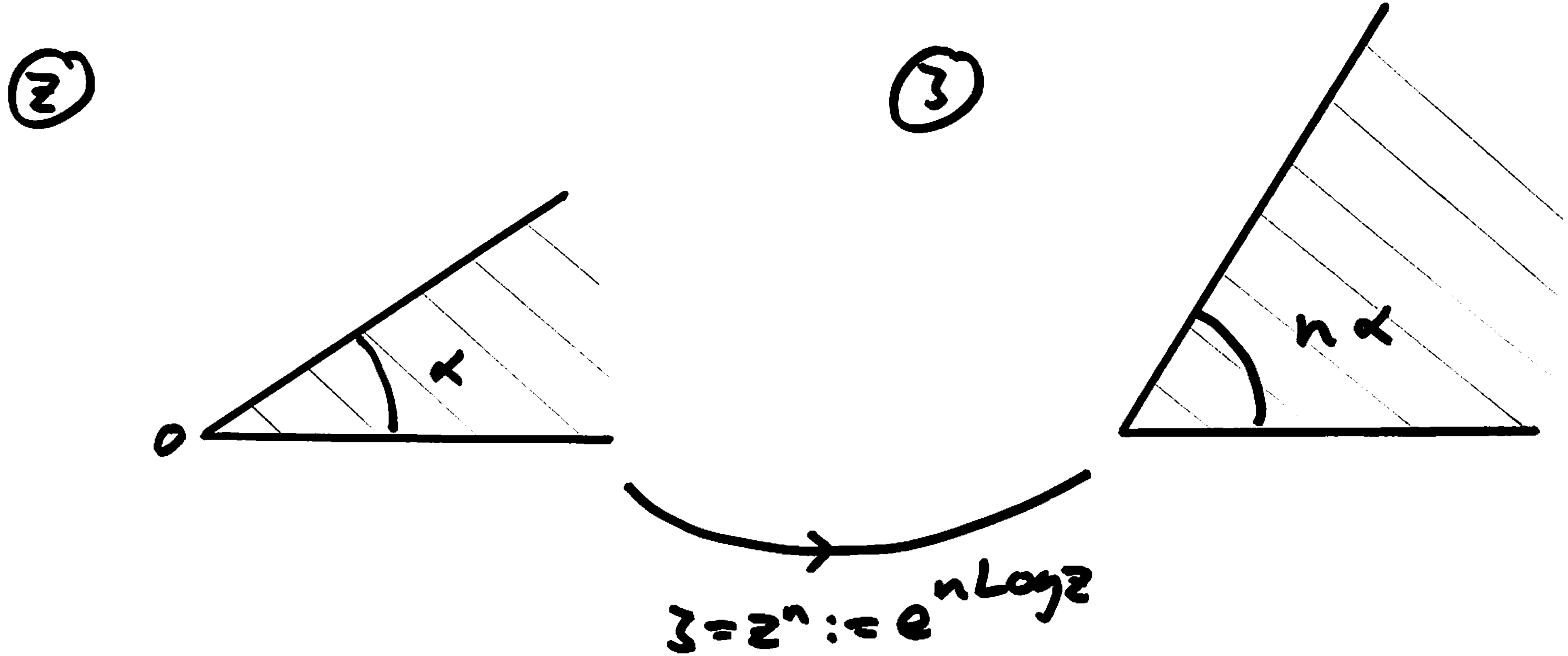
$$\infty \to 0$$

$$i \to \infty$$



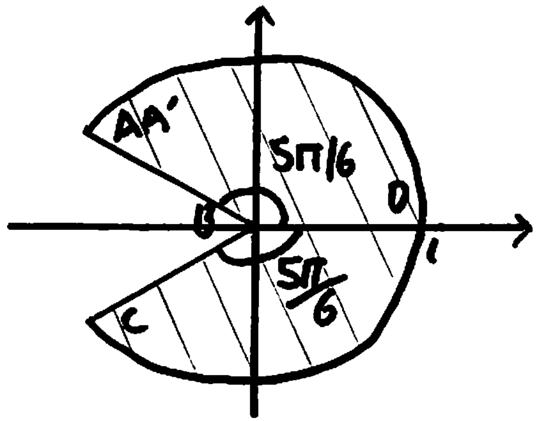
Powers of z

- Use to get rid of corners.
- E.g.



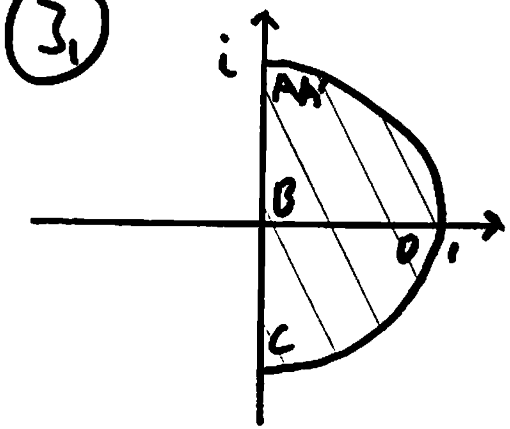
Example

z



$$\zeta_1 = z^{3/5} := e^{\frac{3}{5}\operatorname{Log} z}$$

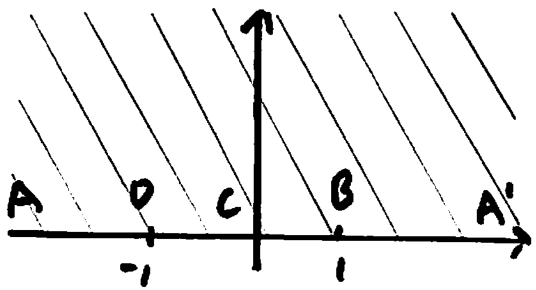
ζ_1



$$\zeta = \left(-\frac{z^{3/5}+i}{z^{3/5}-i}\right)^2$$

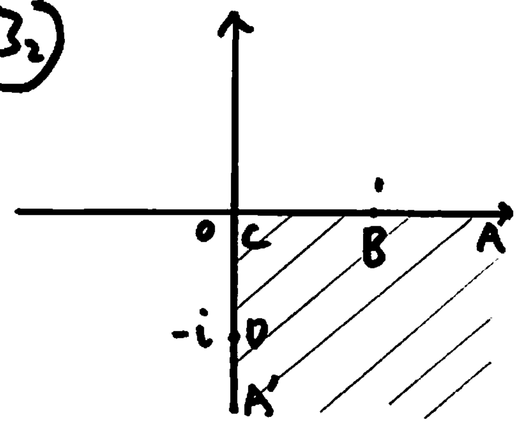
$$\zeta_2 = -\frac{\zeta_1+i}{\zeta_1-i}$$

ζ



$$\zeta = \zeta_2^{\,2}$$

ζ_2



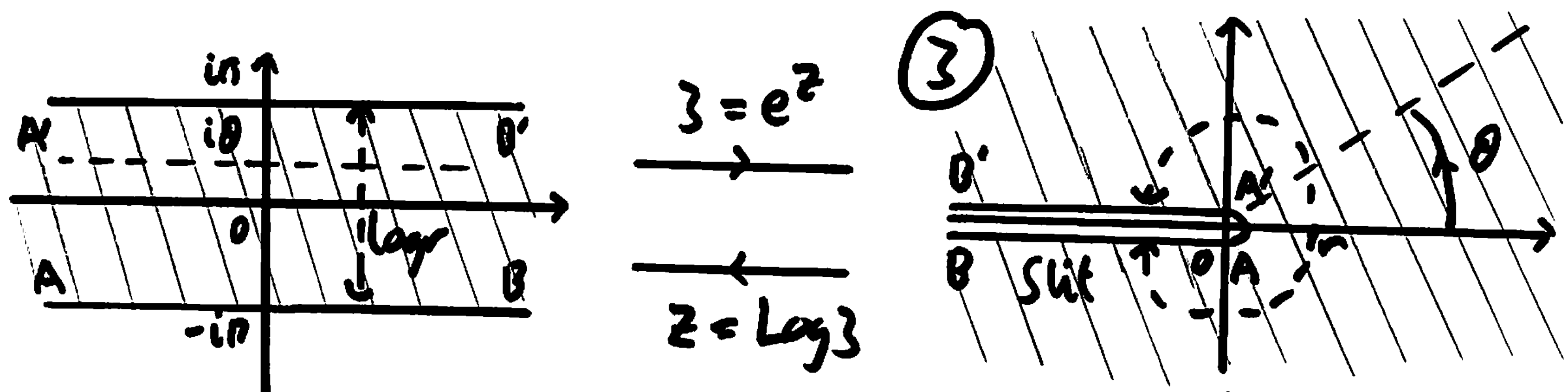
Exponential and logarithm

- Use exp to open a strip or half-strip; log for the reverse.

- E.g. $\zeta = e^z$, $\zeta = re^{i\theta}$, $z = x + iy$

$$\Rightarrow \quad \underbrace{\underbrace{r = e^x}_{\substack{x = \text{constant} \\ \text{are circles}}}, \quad \underbrace{\theta = y}_{\substack{y = \text{constant} \\ \text{are rays}}}}_{\text{exp map generates polar coords.}} \quad \underbrace{(\text{mod } 2\pi)}_{\substack{e^z \text{ is } 2\pi\text{-periodic} \\ \text{in } y.}}$$

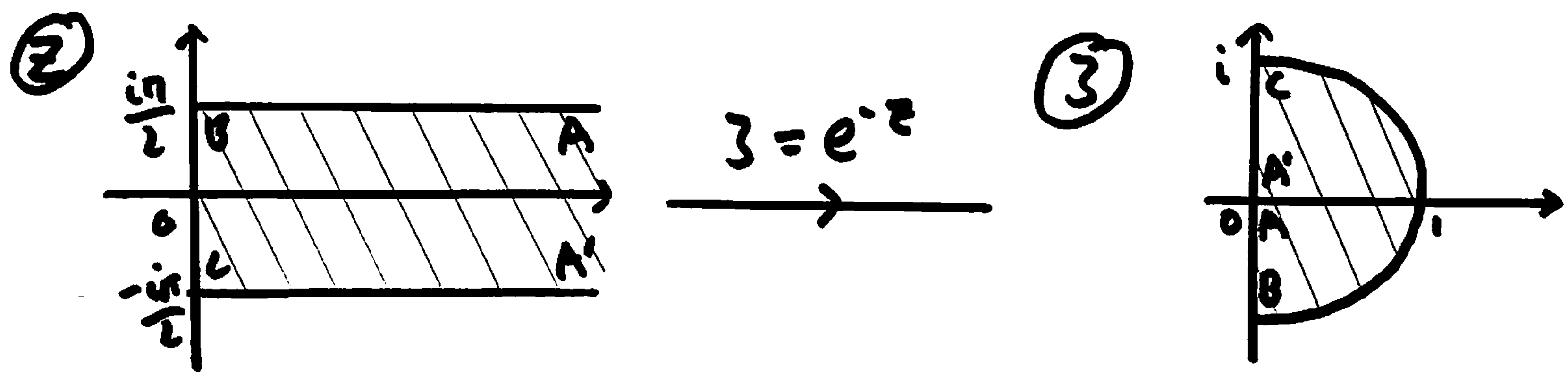
- E.g. z



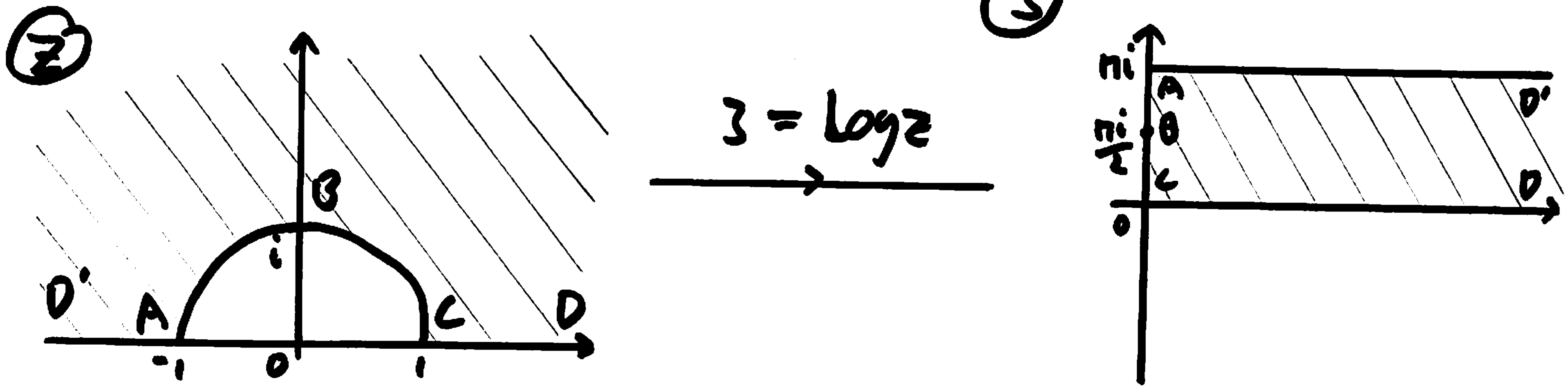
$|\mathrm{Im}(z)| < \pi$ $\qquad$ $|\arg(\zeta)| < \pi$

NB: Same image if strip translated by $2\pi k$ ($k \in \mathbb{Z}$) in y-direction, each one corresponding to a branch of $\log z$.

• E.g.



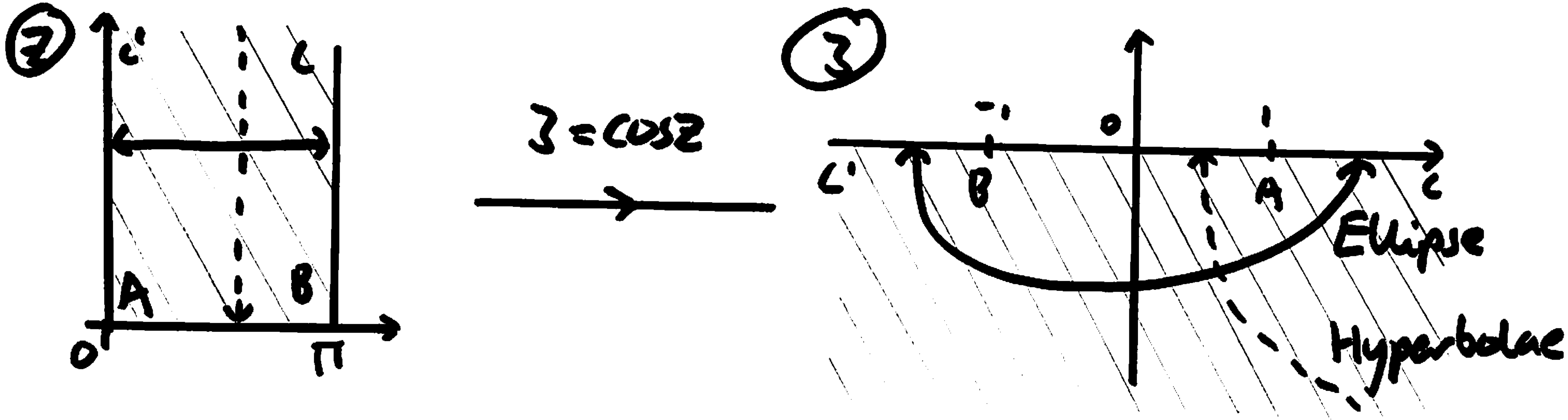
• E.g.



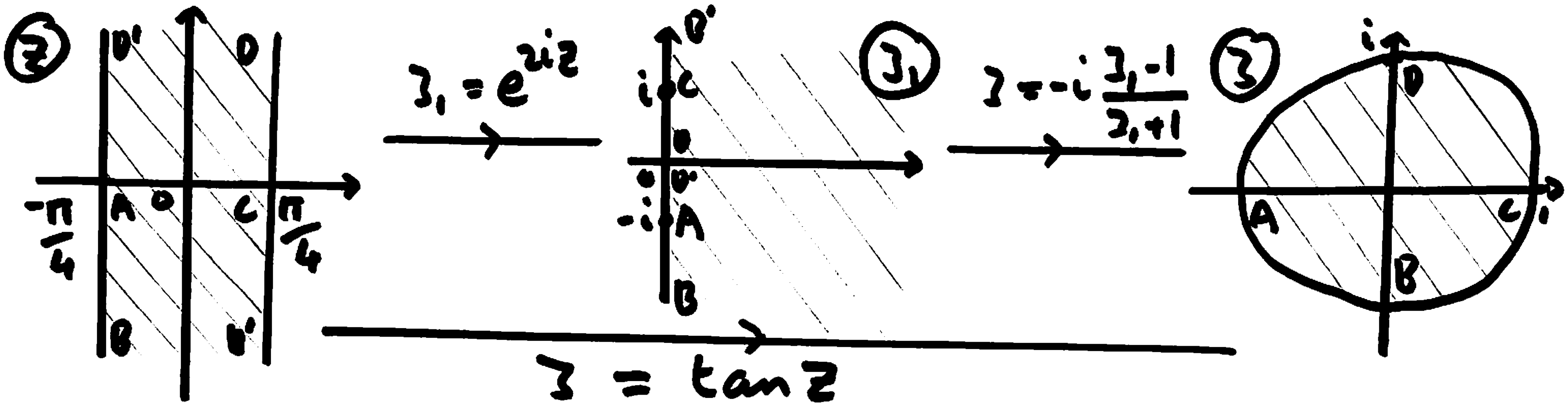
Sin, cos, tan

- Critical points periodically in x-direction $\Rightarrow$ combine angle-doubling and stripmapping properties of z^2 and e^z, resp.

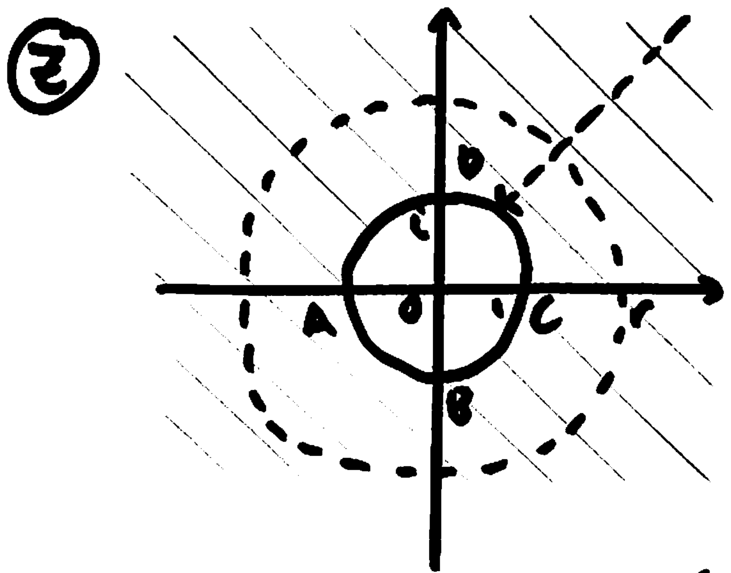
- E.g.

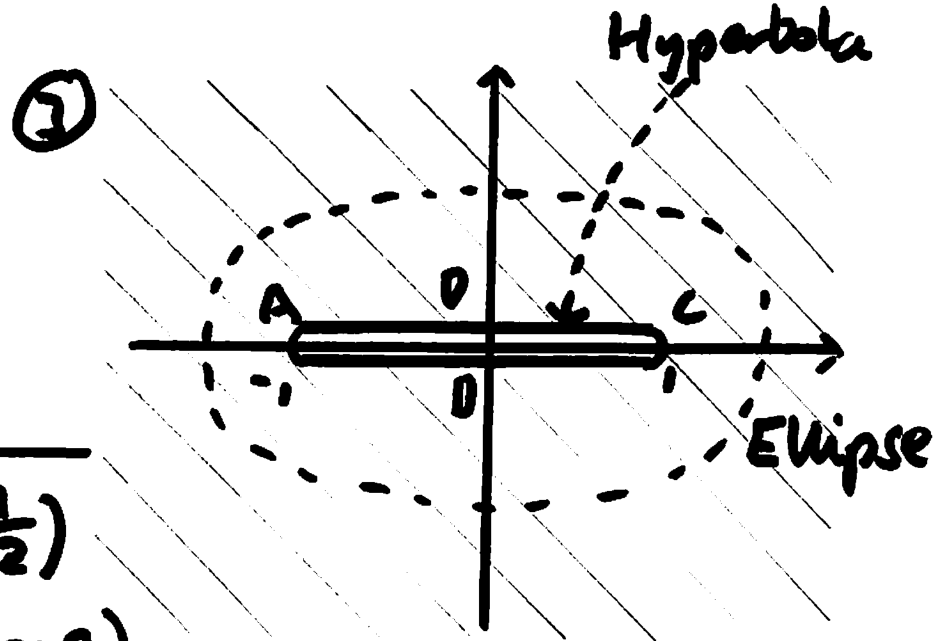


- E.g.



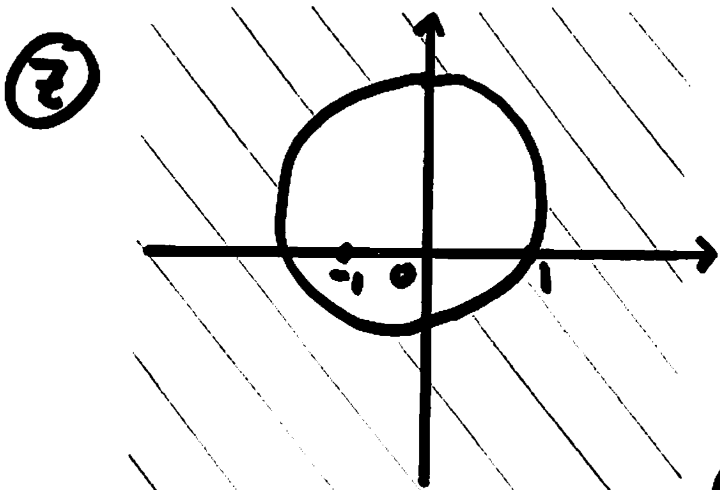
The Joukowski map

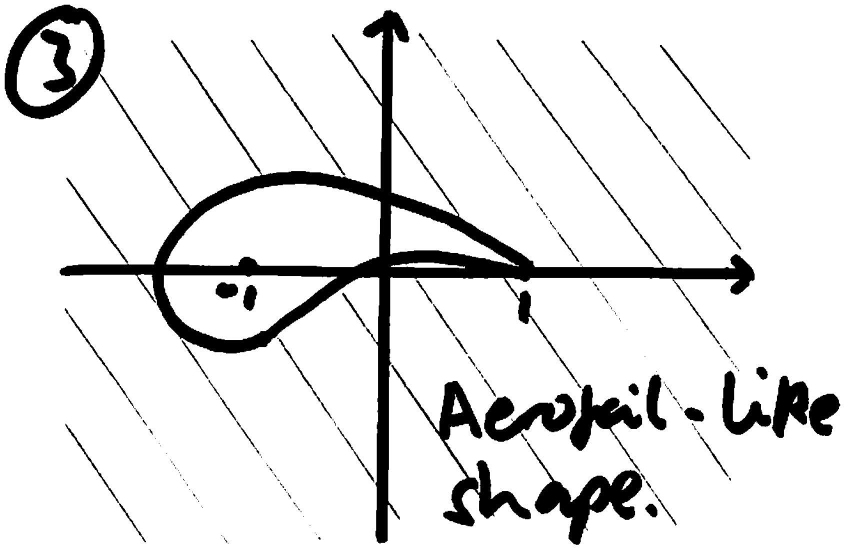




$$\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$(z = e^{i\theta} \Rightarrow \zeta = \cos\theta)$





$$\zeta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

(critical points at $z = \pm 1$)