C4.1 Further Functional Analysis

Sheet 4 — MT 2025

For classes in week 1 HT

Sections A and B are based on material up to and including Section 10. Section C contains some questions based on material from Sections 12 and 13.

Section A

1. Let X and Y be normed spaces $T \in \mathcal{B}(X, Y)$. Fill in the details required to show that T is compact if and only if for every bounded sequence $(x_n)_{n=1}^{\infty}$, there is a subsequence $(x_{n_k})_k$ such that $(Tx_{n_k})_k$ converges.

   Solution: If T is compact, and x_n is a sequence bounded by M, then $T(x_n/M)$ is a sequence in the compact metric space $\overline{T(B_X)}$ so has a convergent subsequence, say $T(x_{n_k}/M)$. Hence $T(x_{n_k})_k$ converges.

   Conversely, if the condition holds, then given a sequence $(y_n) \in \overline{T(B_X)}$, choose $x_n \in B_X$ with $\|Tx_n - y_n\| < 1/n$. Then x_n has a subsequence (x_{n_k}) so that $T(x_{n_k})$ converges, and hence too (y_{n_k}) converges. Thus $\overline{T(B_X)}$ is sequentially compact, so compact (as it is a metric space).

2. Show that c_0 embeds isometrically into $\mathcal{K}(\ell^2)$. Deduce that $\mathcal{K}(\ell^2)$ is not reflexive.

   Solution: For $(a_n)_n \in c_0$ define $T \in \mathcal{B}(\ell^2)$ by $T(x_n) = (a_n x_n)$. It is not difficult to confirm that this is an isometry (the bound from above is immediate; then find an element where the supremum is achieved). One may then define a finite-rank approximation of T via $T^N(x_n) = (a_n x_n)$ for all $n \leq N$, and $T^N(x_n) = 0$ for all $n > N$. Choosing $x \in B_{\ell^2}$ we have

   $$\|T - T^N\| \leq \max_{n \geq N+1} |a_n| \to 0 \text{ as } N \to \infty.$$

   By Theorem 9.4, T maps into the compact operators. Since reflexivity passes to closed subspaces (Theorem 6.8; see also Sheet 2 problem B.7), and c_0 is not reflexive, neither is $\mathcal{K}(\ell^2)$.

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3. This question aims to revise your knowledge of the spectrum of self-adjoint operators on a Hilbert space. If you've not seen this before, then the later parts won't be warm up exercises. Let X be a Hilbert space and $T \in \mathcal{B}(X)$.
   (a) Show that if T is self-adjoint, all eigenvectors are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.
   (b) Show that T is surjective if and only if the adjoint $T^\star$ is bounded below. Use this to show that if $\lambda \in \sigma(T)$ then there is a sequence $(x_n)_{n=1}^\infty$ in S_X such that $(x_n, Tx_n) \to \lambda$.
   (c) If T is self-adjoint show that $\|T\| = \sup_{x \in S_X} |(x, Tx)|$ and deduce that $r(T) = \|T\|$.

Solution:

(a) This is as it was in prelims / part A! Let x be an eigenvector with eigenvalue λ. Then

$$\overline{\lambda}\|x\|^2 = \langle Tx, x\rangle = \langle x, Tx\rangle = \lambda\|x\|^2.$$

Since $x \neq 0$, $\overline{\lambda} = \lambda$, and $\lambda \in \mathbb{R}$.

Suppose $y \in X$ is an eigenvector with eigenvalues μ. As

$$\lambda\langle x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \mu\langle x, y\rangle,$$

if $\lambda \neq \mu$, then $\langle x, y\rangle = 0$.

(b) Suppose T is surjective, then T is a quotient operator so T^* is an isomorphic embedding so bounded below. Hence $T^\star$ is also bounded below (from the connection between the dual operator $T^* : H^* \to H^*$ and the Hilbert space adjoint $T^\star : H \to H$.) Conversely if $T^\star$ is bounded below, then it is an isomorphic embedding and hence so too is T^*. Then T is a quotient operator, so surjective.

Suppose $\lambda \in \sigma(T)$. Either $T - \lambda I$ is not injective, when there exists $x \in S_X$ with $Tx = \lambda x$, or $T - \lambda I$ is not surjective. Thus $(T - \lambda I)^\star = T^\star - \overline{\lambda} I$ is not bounded below, and hence there exists a sequence $(x_n)_n$ in S_X with $\|(T - \lambda I)^\star x_n\| \to 0$. Thus $\langle T^\star x_n - \overline{\lambda} x_n, x_n\rangle \to 0$ and so $\langle x_n, Tx_n\rangle \to \lambda$.

(c) Recall that for $T \in \mathcal{B}(X)$, we have $\|T\| = \sup_{x,y \in S_X} |\langle Tx, y\rangle|$.

Now, for $T = T^\star$, note that for $x, y \in S_X$,

$$\begin{aligned}\langle T(x+y), x+y\rangle - \langle T(x-y), (x-y)\rangle = 2\langle Tx, y\rangle + 2\langle Ty, x\rangle &= 2\langle Tx, y\rangle + 2\langle x, Ty\rangle \\ &= 4\Re\langle Tx, y\rangle.\end{aligned}$$

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Now, writing $M = \sup_{x \in S_X} |\langle Tx, x\rangle|$, we have

$$\begin{aligned}\Re\langle Tx, y\rangle &= \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), (x-y)\rangle) \\ &\le \frac{M}{4}(\|x+y\|^2 + \|x-y\|^2) = \frac{M}{4}(2\|x\|^2 + \|y\|^2) \le M,\end{aligned}$$

using the parallelogram law. Multiplying by a suitable scalar of norm 1 we get $|\langle Tx, y\rangle| \le M$.

We have $r(T) \le \|T\|$ for all bounded operators T. Now let $T = T^\star$ and choose a sequence $x_n \in S_X$ with $|\langle Tx_n, x_n\rangle| \to \|T\|$. We can pass to a subsequence such that $\langle Tx_n, x_n\rangle \to \lambda = \pm\|T\|$. Now (as $\langle Tx_n, x_n\rangle, \lambda$ are both real),

$$\begin{aligned}0 \le \|(T-\lambda)x_n\|^2 &= \langle (T-\lambda)x_n, (T-\lambda)x_n\rangle \\ &= \|Tx_n\|^2 - 2\lambda\langle Tx_n, x_n\rangle + \lambda^2 \le 2\lambda^2 - 2\lambda\langle Tx_n, x_n\rangle \to 0.\end{aligned}$$

Hence $T - \lambda I$ is not bounded below, and hence $\lambda = \pm\|T\| \in \sigma(T)$. Thus $r(T) \le \sigma(T)$.

4. Let X be a separable Hilbert space. An operator $T \in \mathcal{B}(X)$ is a *Hilbert-Schmidt* operator if there is an orthonormal basis $(e_n)_{n=1}^\infty$ for X such that $\sum_{n=1}^\infty \|T(e_n)\|^2 < \infty$.

   (a) Show that if $(e_n)_{n=1}^\infty$ and $(f_m)_{m=1}^\infty$ are orthonormal bases for X, then $\sum_m \|T(f_m)\|^2 = \sum_n \|T(e_n)\|^2$ for any $T \in \mathcal{B}(X)$.

   Solution: For each $m \in \mathbb{N}$, write $f_m = \sum_n \alpha_n^m e_n$, where $\alpha_n^m \in \mathbb{F}$ is zero for all but finitely many values of n. Then

$$\begin{aligned}\sum_m \|Tf_m\|^2 &= \sum_m \left\| T\left(\sum_n \alpha_n^m e_n\right)\right\|^2 \\ &= \sum_m \left\langle \sum_i \alpha_i^m Te_i, \sum_j \alpha_j^m Te_j \right\rangle \\ &= \sum_m \sum_{i,j} \alpha_i^m \overline{\alpha_j^m}\, \langle Te_i, Te_j\rangle \\ &= \sum_{i,j} \langle Te_i, Te_j\rangle \sum_m \alpha_i^m \overline{\alpha_j^m} \\ &= \sum_{i,j} \langle Te_i, Te_j\rangle \sum_m \alpha_i^m \overline{\alpha_j^m}.\end{aligned}$$

   We now prove that $\sum_m \alpha_i^m \overline{\alpha_j^m} = \delta_{ij}$, which will conclude the proof of this part. Write $e_n = \sum_m \beta_m^n f_m$. Then $\langle e_n, f_m\rangle = \beta_m^n = \alpha_n^m$, so

$$\sum_m \alpha_i^m \overline{\alpha_j^m} = \sum_m \beta_m^i \overline{\beta_m^j} = \langle e_i, e_j\rangle = \delta_{ij},$$

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as desired.

(b) Show that every Hilbert-Schmidt operator is compact.

Solution: Consider the sequence of operators $T_n \in \mathcal{B}(X)$ defined by $T_n(e_i) = Te_i$ for $i \leqslant n$ and $T_n(e_i) = 0$ for $i > n$ and extend to all of X by density. Note that $T_n(X)$ is the closure of $T_n|_{\operatorname{span}\{e_i\}}(X)$. But $T_n|_{\operatorname{span}\{e_i\}}(X)$ is finite dimensional and therefore closed. Thus, the operators T_n have finite rank. We now show that $\lim_{n\to\infty} \|T - T_n\| = 0$. Let ε and let $n \in \mathbb{N}$ be such that $\sum_{i>n} \|Te_i\|^2 < \varepsilon$, which is possible since T is Hilbert–Schmidt. Let $x = \sum_i \alpha_i e_i \in B_X$. Then

$$\|(T - T_n)(x)\|^2 = \left\|\sum_{i>n} \alpha_i Te_i\right\|^2 \leqslant \sum_{i>n} |\alpha_i|^2 \cdot \|Te_i\|^2 \leqslant \sum_{i>n} \|Te_i\|^2 < \varepsilon.$$

This proves the claim and shows that T is compact by Corollary 11.4 of the notes.

(c) Give a characterisation in terms of eigenvalues and multiplicities of when a compact self-adjoint operator is Hilbert-Schmidt.

Solution: Let T be a compact self-adjoint operator. By the Spectral Theorem, there are nonzero real numbers λ_i and finite-rank orthogonal projections P_i such that $T = \sum_i \lambda_i P_i$ (the sum is either finite or countably infinite) and there is a basis E of orthonormal eigenvectors. For each i, let $e_1^i, \dots, e_{m(i)}^i$ be the eigenvectors of E with eigenvalue to λ_i. Note that $m(i)$ is the multiplicity of λ_i, and it is always finite (by Theorem 13.3, for example).

$$\begin{aligned}
\sum_{e\in E} \|Te\|^2 &= \sum_i \sum_{k=1}^{m(i)} \left\|Te_k^i\right\|^2 \\
&= \sum_i \sum_{k=1}^{m(i)} \left\langle \sum_j \lambda_j P_j e_k^i, \sum_j \lambda_j P_j e_k^i \right\rangle \\
&= \sum_i \sum_{k=1}^{m(i)} |\lambda_i|^2 \left\langle \sum_{k=1}^{m(i)} e_k^i, \sum_{k=1}^{m(i)} e_k^i \right\rangle \\
&= \sum_i \sum_{k=1}^{m(i)} |\lambda_i|^2 m(i) \\
&= \sum_i |\lambda_i|^2 m(i)^2.
\end{aligned}$$

We thus obtain the following characterisation: T is Hilbert–Schmidt if and only if

$$\sum_i |\lambda_i|^2 m(i)^2 < \infty.$$

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Section B

1. (a) Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X, Y)$. We say that T is *completely continuous* if, for every weakly convergent sequence (x_n) in X, the sequence (Tx_n) is norm-convergent in Y.
   (i) Show that if T is compact then T is completely continuous.
   (ii) Prove that the converse of (i) holds if X is reflexive. [*You may, if you wish, assume in addition that X is separable.*]
   (iii) Exhibit an operator which is completely continuous but not compact.

   (b) Let $1 < p < \infty$. Show that $\mathcal{B}(\ell^p, \ell^1) = \mathcal{K}(\ell^p, \ell^1)$. Is $\mathcal{B}(c_0, \ell^p) = \mathcal{K}(c_0, \ell^p)$?

2. Let $K \in L^2(\mathbb{R}^2)$ and consider the map T sending $x \in L^2(\mathbb{R})$ to the function Tx defined by
$$(Tx)(t) = \int_{\mathbb{R}} K(s,t)x(s)\,\mathrm{d}\,s$$
whenever $t \in \mathbb{R}$ is such that the integral exists.
   (a) Show that T is a well-defined element of $\mathcal{B}(L^2(\mathbb{R}))$ with $\|T\| \leq \|K\|_{L^2(\mathbb{R}^2)}$.
   (b) Prove that T is compact. [*You may use the fact that indicator functions of bounded rectangles span a dense subspace of $L^2(\mathbb{R}^2)$.*]

3. Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X, Y)$. We say that T is *weakly compact* if the weak closure of $T(B_X)$ is weakly compact.
   (a) Show that T is weakly compact if and only if $\operatorname{Ran} T^{**} \subseteq J_Y(Y)$.
   (b) Prove that if T is weakly compact then T^* is weakly compact, and that if Y is complete then the converse holds too.

4. Let X, Y be Banach spaces and suppose that $T \in \mathcal{B}(X, Y)$. Show that T is Fredholm if and only if T^* is and that, if both operators are Fredholm, then $\operatorname{ind} T + \operatorname{ind} T^* = 0$.

5. Let X, Y and Z be Banach spaces and let $S \in \mathcal{B}(Y, Z)$ and $T \in \mathcal{B}(X, Y)$.
   (a) Show that if S, T are both Fredholm then so is ST and $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$.
   (b) Suppose now that ST is Fredholm. Prove that S is Fredholm if and only if T is Fredholm. Give an example in which neither S nor T is Fredholm.
   (c) Show that if $X = Y = Z$ and $ST = TS$ then ST is Fredholm if and only if S and T are both Fredholm.

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Section C

1. Let X be the complex Banach space ℓ^1 and consider the left-shift operator $T \in \mathcal{B}(X)$ given by $Tx = (x_{n+1})_{n\geq 1}$ for $x = (x_n)_{n\geq 1} \in X$. Moreover let $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

   (a) Show that for $\lambda \in \mathbb{C}$ the operator $T - \lambda$ is Fredholm if and only if $\lambda \notin \Gamma$, and determine the index $\operatorname{ind}(T - \lambda)$ whenever it is defined.

   (b) Let p be a complex polynomial. Prove that $p(T)$ is Fredholm if and only if $p^{-1}(\{0\}) \cap \Gamma = \emptyset$ and that, if this condition is satisfied, then

$$\operatorname{ind} p(T) = \frac{1}{2\pi i} \oint_\Gamma \frac{p'(\lambda)}{p(\lambda)} \,\mathrm{d}\,\lambda.$$

Solution:

(a) Since $\|T\| = 1$ we know that for $|\lambda| > 1$ the operator $T - \lambda$ is invertible and hence Fredholm with $\operatorname{ind}(T - \lambda) = 0$. If $|\lambda| < 1$ then it is easy to verify that $\operatorname{Ker}(T - \lambda) = \operatorname{Span}\{(1, \lambda, \lambda^2, \dots)\}$. We now show that $T - \lambda$ is surjective for $|\lambda| < 1$. This can be done directly, or alternatively by noting that if we identify X^* with ℓ^∞ then T^* is the right shift defined by $T^*x = (0, x_1, x_2, \dots)$ for $x = (x_n) \in \ell^\infty$. In particular, T^* is an isometry and $\|T^*x - \lambda x\| \geq (1 - |\lambda|)\|x\|$ for all $x \in \ell^\infty$, so $T^* - \lambda$ is an isomorphic embedding. Since X is complete it follows from a result in lectures that $T - \lambda$ is a quotient operator and in particular surjective. Hence $T - \lambda$ is Fredholm and $\operatorname{ind}(T - \lambda) = 1$ for $|\lambda| < 1$. Since the index is locally constant on the set of Fredholm operators, $T - \lambda$ cannot be Fredholm when $|\lambda| = 1$.

(b) Let p be a complex polynomial. If p is constant then $p(T)$ is Fredholm if and only if p is non-zero. If p is non-constant then we may write it in the form $p(\lambda) = c(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$ for some $n \geq 1$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \{0\}$. Then

$$p(T) = c \prod_{k=1}^{n} (T - \lambda_k),$$

and by a previous result and induction we obtain that $p(T)$ is Fredholm if and only if $T - \lambda_k$ is Fredholm for $1 \leq k \leq n$. By part (a) we know that $T - \lambda_k$ is Fredholm if and only if $|\lambda_k| \neq 1$, so $p(T)$ is Fredholm if and only if $\{\lambda_1, \dots, \lambda_n\} \cap \Gamma = \emptyset$. But $\{\lambda_1, \dots, \lambda_n\} = p^{-1}(\{0\})$. Thus in all cases $p(T)$ is Fredholm if and only if $p^{-1}(\{0\}) \cap \Gamma = \emptyset$. Moreover, if $p(T)$ is Fredholm then by Q.2 and the first part

$$\operatorname{ind} p(T) = \sum_{k=1}^{n} \operatorname{ind}(T - \lambda_k) = \frac{1}{2\pi i} \sum_{k=1}^{n} \oint_\Gamma \frac{\mathrm{d}\lambda}{\lambda - \lambda_k} = \frac{1}{2\pi i} \oint_\Gamma \frac{p'(\lambda)}{p(\lambda)} \,\mathrm{d}\lambda.$$

Remark: Suppose that p has no zeros on the unit circle Γ, so that $p(T)$ is Fredholm. Let $\Gamma_p = p(\Gamma)$ and parameterise Γ_p (with the orientation inherited from Γ being

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traversed counterclockwise) by $t \mapsto r(t)e^{i\theta(t)}$, where $r(t) > 0$ and $\theta(t)$ is real for $0 \le t \le 1$. Then $r(0) = r(1)$ and hence

$$\operatorname{ind} p(T) = \frac{1}{2\pi i}\oint_{\Gamma_p} \frac{\mathrm{d}\lambda}{\lambda} = \frac{1}{2\pi i}\int_0^1 \left(\frac{r'(t)}{r(t)} + i\theta'(t)\right)\mathrm{d}t = \frac{\theta(1)-\theta(0)}{2\pi}.$$

Thus $\operatorname{ind} p(T)$ coincides with the *winding number* of the curve Γ_p, that is to say its total number of counterclockwise revolutions about the origin. There are many other situations in which the Fredholm index has a topological interpretation. For instance, the famous *Atiyah-Singer Index Theorem* states that for elliptic differential operators on compact manifolds the Fredholm index equals a certain topological index.

2. Let X be a Banach space and let $\{x_n : n \ge 1\}$ be a Schauder basis for X with basis projections P_n, $n \ge 1$, and let

$$|||x||| = \sup\{\|P_n x\| : n \ge 1\}, \quad x \in X.$$

Prove that $|||\cdot|||$ defines a complete norm on X.

Solution: If $|||x||| = 0$, then $\|P_n x\| = 0$ for all $n \in \mathbb{N}$ and therefore the representation of x in terms of the Schauder basis is $x = \sum_n 0 \cdot x_n$. Hence $x = 0$. Next,

$$|||\lambda x||| = \sup_{n\in\mathbb{N}}\{\|P_n(\lambda x)\|\} = \sup_{n\in\mathbb{N}}\{|\lambda| \cdot \|P_n x\|\} = |\lambda| \cdot \sup_{n\in\mathbb{N}}\{\|Px_n\|\} = |\lambda| \cdot |||x|||$$

for all $\lambda \in \mathbb{F}$. For the triangle inequality, we have

$$\begin{aligned} |||x+y||| &= \sup_{n\in\mathbb{N}}\{\|P_n(x+y)\|\} \\ &= \sup_{n\in\mathbb{N}}\{\|P_n x + P_n y\|\} \\ &\leqslant \sup_{n\in\mathbb{N}}\{\|P_n x\| + \|P_n y\|\} \\ &\leqslant \sup_{n\in\mathbb{N}}\{\|P_n x\|\} + \sup_{n\in\mathbb{N}}\{\|P_n y\|\} \\ &= |||x||| + |||y||| \end{aligned}$$

for all $x, y \in X$.

Now we verify that $|||\cdot|||$ is complete. Let $(a_i)_{i\in\mathbb{N}} = (\alpha_1^i x_1 + \alpha_2^i x_2 + \cdots)_{i\in\mathbb{N}}$ be a Cauchy sequence in $(X, |||\cdot|||)$. Let $\varepsilon > 0$. Then there is some $N \in \mathbb{N}$ such that for all $i, j > N$ we have

$$\begin{aligned} \varepsilon > |||a_i - a_j||| &= \sup_{n\in\mathbb{N}}\left\{\left\|(\alpha_1^i - \alpha_1^j)x_1 + \cdots + (\alpha_n^i - \alpha_n^j)x_n\right\|\right\} \\ &\geqslant \left\|(\alpha_1^i - \alpha_1^j)x_1 + \cdots + (\alpha_n^i - \alpha_n^j)x_n\right\| \end{aligned}$$

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for every $n \in \mathbb{N}$. In particular, this implies that $(\alpha_1^i x_1 + \cdots + \alpha_n^i x_n)_i$ is a Cauchy sequence for each $n \in \mathbb{N}$. It is then easy to show that each $(\alpha_n^i)_i$ is a Cauchy sequence for every $n \in \mathbb{N}$, and therefore that there are constants $\alpha_1, \alpha_2, \cdots$ such that

$$\lim_{i\to\infty}(\alpha_1^i x_1 + \cdots + \alpha_n^i x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n$$

for each $n \in \mathbb{N}$. Formally define $a = \sum_n \alpha_n x_n$; we have not yet shown that the series on the right hand side is norm convergent, but for now we will only be interested in the projections $P_n a = \alpha_1 x_1 + \cdots + \alpha_n x_n$ which are well-defined.

Let $n \in \mathbb{N}$, let $i > N$, and let $j > N$ be such that $\|P_n a - P_n a_j\| < \varepsilon$. Then

$$\|P_n a - P_n a_i\| \leqslant \|P_n a - P_n a_j\| + \|P_n a_j - P_n a_i\| < 2\varepsilon.$$

Since the above inequality is independent of n, we obtain

$$\sup_{n\in\mathbb{N}}\{\|P_n a - P_n a_i\|\} < 2\varepsilon$$

for all $i > N$. It only remains to show that the series defining a is in fact norm convergent so that a is actually a well-defined element of X. To do this, we will show that $(P_n a)_{n\in\mathbb{N}}$ isa Cauchy sequence in $(X, \|\cdot\|)$. Let $m, n > N$. Then for every $i > N$ we have

$$\|P_m a - P_n a\| \leqslant \|P_m a - P_m a_i\| + \|P_m a_i - P_n a_i\| + \|P_n a_i - P_n a\| < 4\varepsilon + \|P_m a_i - P_n a_i\|$$

But $(P_n a_i)$ is Cauchy for every i, so there is some N_i such that $\|P_m a_i - P_n a_i\| < \varepsilon$ for all $m, n > N_i$. So for $m, n > \max\{N, N_i\}$, we obtain $\|P_m a - P_n a\| < 5\varepsilon$, which proves the claim.

3. (a) Prove that if X is a Banach space with a Schauder basis, then every compact operator on X is a norm limit of finite rank operators. [1]
   (b) Show that if $T : X \to Y$ is a finite rank operator, then so too is T^*.
   (c) Suppose that X is a Banach space with a Schauder basis. Show how to use parts (a) and (b) above to deduce that if $T : X \to Y$ is compact, then so too is T^* in this case. Note that this result applies when X is a Hilbert space.

[1] Additional exercise. Show that regardless of separability, every compact operator on a Hilbert space is a limit of finite rank operators.

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