B4.3: Distribution Theory

Hilary Term

20 January - 13 March 2025 (16 hours)

Tuesdays: 16:00 - 16:55 (L5)
Fridays: 10:00 - 11:55 (L6)

Lecturer: Prof. Gui-Qiang G. Chen
Tutors: Aidan Strong and Prof. Qian Wang

Overview

- **Distribution Theory can be thought of as the completion of Differential Calculus**, just as Lebesgue Integration Theory can be thought of as the completion of Integral Calculus.
- **Distribution Theory was created by Laurent Schwartz in the 20th century,** not long after Lebesgue's Integration Theory.
- Besides being an important part of Analysis, it has many applications. One of the main areas of applications is:

  Theory of Partial Differential Equations

  and a brief treatment, mainly through examples, is included in this course and its continuation ***B4.4: Fourier* Analysis (*Reading Course*).**
- A more systematic study is deferred to the Part C/OMMS courses ***Functional Analytic Methods for PDEs***, ***Fixed Point Methods for Nonlinear PDEs,*** and ***Optimal Transport and PDEs***.
- The course also provides preparation for many other Part C/OMMS courses, including ***Analytic Number Theory***, ***Further Functional Analysis***, ***Topics in Fluid Mechanics***, ***Applied Complex Variables***, ***Stochastic Analysis and PDEs, and other PDE Courses***.

General Prerequisites

- **Part A: *Integration*** is essential.
- A good working knowledge of **Part A: *Metric Spaces and Complex Analysis*** is expected.
- **Part A: *Integral Transforms* and *Multidimensional Analysis and Geometry*** are desirable but not essential.

Learning Outcomes

You will become acquainted with the basic techniques that, in many situations, form the starting point for the modern treatment of

Partial Differential Equations (PDEs).

Core Reading

R.S. Strichartz: A Guide to Distribution Theory and Fourier Transforms, World Scientific, 1994. Reprinted: 2008, 2015. In particular, Chapters 1, 2, and 6.

Further Reading

- **L.C. Evans: Partial Differential Equations,** 2nd Ed., Graduate Studies in Mathematics, 19, American Mathematical Society, 2010
- **E.H. Lieb and M. Loss: Analysis,** 2nd Ed., Graduate Studies in Mathematics, American Mathematical Society, 2001
- **E.M. Stein and R. Shakarchi: Fourier Analysis: An Introduction,** Princeton Lectures in Analysis, III. Princeton University Press, Princeton, NJ, 2003

Synopsis:

1. **Test functions and distributions on R^n**: Definitions and examples, Dirac δ- function, Approximate identities and constructions using convolution of functions. **Density of test functions in Lebesgue spaces**. **Smooth partitions of unity**. [4 lectures]

2. **The calculus of distributions on R^n** : Functions as distributions, Operations on distributions, Adjoint identities, Consistency of derivatives, Convolution of test functions and distributions. **The Fundamental Theorem of Calculus for distributions**. **Support and singular support of a distribution**. **Convolution with a compactly supported distribution**. **Examples of distributions** defined by principal value integrals and finite parts. Examples of distributional boundary values of holomorphic functions defined in a half-plane. [8 lectures]

3. **Distributional and weak solutions of PDEs**, Absolutely continuous functions, Sobolev functions. **Examples of fundamental solutions**. **Weyl's Lemma for distributions**. **Convolution rules for support and singular support.** [4 lectures]