Problem sheet 1

1. Given n positive numbers $x_1, x_2, \ldots, x_n$ such that $x_1 + x_2 + \cdots + x_n \leqslant 1/3$, prove by induction that
$$(1 - x_1)(1 - x_2) \times \cdots \times (1 - x_n) \geqslant 2/3.$$
[*Hint: for the inductive step, consider* $x_1, x_2, \ldots, x_{n-1}, x_n + x_{n+1}$.]

2. Using the recursive definition of addition given in lectures, follow the steps below to show that addition on $\mathbb{N}$ is commutative; that is, $x + y = y + x$ for all $x, y \in \mathbb{N}$.
   (i) First prove the result for $y = 0$ and all x by inducting on x.
   (ii) Next prove the result for $y = 1$ and all x.
   (iii) Prove the general result by inducting on y.

3. (i) Let A, B, and C be subsets of a set S. Write out a proof that
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)\,.$$
   (ii) Let D, E, and F be subsets of a set S. Use (i) and De Morgan's laws to show that
$$D \cup (E \cap F) = (D \cup E) \cap (D \cup F)\,.$$

4. Let A and B be finite sets.
   (i) Assuming for this part that A and B are disjoint, and adopting the recursive definition of cardinality given in lectures, use induction on $|B|$ to show that $A \cup B$ is finite and that
$$|A \cup B| = |A| + |B|\,.$$
   (ii) Show that for general A and B, $A \cup B = (A \backslash B) \cup (A \cap B) \cup (B \backslash A)$. Deduce that
$$|A| + |B| = |A \cup B| + |A \cap B|\,.$$

5. Which of the following relations on $\mathbb{N}$ are reflexive, which are symmetric, which are transitive?
   (i) the relation $a|b$ (read as 'a divides b').
   (ii) the relation $a \nmid b$ (does not divide).
   (iii) a, b are related if a, b leave the same remainder after division by 2021.
   (iv) a, b are related if $\mathrm{hcf}(a, b) > 2021$.
   [$\mathrm{hcf}(a, b)$ *denotes the highest common factor, or greatest common divisor, of* a *and* b. *It is the largest natural number that divides both* a *and* b.]

6. How many partitions are there of a set of size 1? of size 2? of size 3? of size 4? of size 5?

7. Let $S = \{(m, n) : m, n \in \mathbb{Z},\ n \geqslant 1\}$.
   (i) Show that $\sim$ defined by
$$(m_1, n_1) \sim (m_2, n_2) \quad \text{if and only if} \quad m_1 n_2 = m_2 n_1$$
   is an equivalence relation on S.

(ii) The set of equivalence classes is denoted $S/\sim$. Show that $\oplus$ and $\otimes$ defined by

$$\overline{(m_1, n_1)} \oplus \overline{(m_2, n_2)} = \overline{(m_1 n_2 + m_2 n_1, n_1 n_2)}$$
$$\overline{(m_1, n_1)} \otimes \overline{(m_2, n_2)} = \overline{(m_1 m_2, n_1 n_2)}$$

are well-defined binary operations on $S/\sim$. What set is $S/\sim$ a model for?