Problem sheet 2

1. Define $f\colon \mathbb{R} \to \mathbb{R}$ by $f(x) = \sin x$ for all real numbers x.

   (i) What are $f([0,\pi])$, $f([0,2\pi])$, $f([0,3\pi])$?
   (ii) What are $f^{-1}(\{0\})$, $f^{-1}(\{1\})$, $f^{-1}(\{2\})$?
   (iii) Let $A = [0,\pi]$ and $B = [2\pi, 3\pi]$. Show that $f(A \cap B) \neq f(A) \cap f(B)$.
   (iv) Let $A = [0,\pi]$. Find $f^{-1}(f(A))$ and $f(f^{-1}(A))$.

2. Let $A = \{1, 2, 3\}$ and $B = \{1, 2, 3, 4, 5\}$.

   (i) How many $f\colon A \to B$ are there?
   (ii) How many $f\colon B \to A$ are there?
   (iii) How many injective $f\colon A \to B$ are there?
   (iv) How many injective $f\colon B \to A$ are there?
   (v) How many surjective $f\colon A \to B$ are there?
   (vi) How many surjective $f\colon B \to A$ are there?

3. Which of the following statements about natural numbers are true, which false?

   (i) 2 is prime or 2 is odd.
   (ii) 2 is prime or 2 is even.
   (iii) If 2 is odd then 2 is prime.
   (iv) If 2 is even then 2 is prime.
   (v) For all $n \in \mathbb{N}$, if n is a square number then n is not prime.
   (vi) For all $n \in \mathbb{N}$, n is not prime if and only if n is a square number.
   (vii) For all even primes $p > 2$, $p^2 = 2021$.

4. Organise the following assertions about a function $f : \mathbb{R} \to \mathbb{R}$ into pairs such that one member of the pair is true if and only if the other is false. Re-write each statement using symbols ($\in$, $\forall$, $\exists$, etc.)

   (1) for all real numbers x, y there exists a real number $z > x$ such that $f(z) > y$;
   (2) for every real number x there exists a real number y such that for all real z either $z \leq y$ or $f(z) \neq x$;
   (3) there exist real numbers x, y such that for every real number z if $z \leq y$ then $f(z) = x$;
   (4) there is a real number x such that for every real number y there is a real number $z > y$ for which $f(z) = x$;
   (5) for all real x and y there is a real number z such that $z \leq y$ and $f(z) \neq x$;
   (6) there exist real numbers x and y such that for every real number z either $z \leq x$ or $f(z) \leq y$.

5. Each card in a pack has a number on one side and a letter on the other. Four cards are placed on the table:

   $\boxed{\mathbf{2}}$ $\boxed{\mathbf{3}}$ $\boxed{\mathbf{A}}$ $\boxed{\mathbf{B}}$

   You are permitted to turn just two cards over in order to test the following hypothesis: *a card that has an even number on one side has a vowel on the other.* Which two cards should you turn? Or is it impossible?

6. Let X, Y and Z be sets, and let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions. Which of the following statements are true, which false? In each case either carefully prove the statement or give a specific counter-example.

   (i) if f and g are surjective then $g \circ f$ is surjective.
   (ii) if g is surjective then $g \circ f$ is surjective.
   (iii) if $g \circ f$ is surjective then g is surjective.

7. Let $n \geq 2$ be an integer, let S be the set of $n \times n$ matrices with real coefficients, and define $f : S \to \mathbb{R}$ by

$$f(A) = \sum_{i=1}^{n} a_{ii} \quad \text{for} \quad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}.$$

   Prove or disprove each of the following statements:

   (i) $f(A+B) = f(A) + f(B)$ for all $A, B \in S$;
   (ii) $f(AB) = f(A)f(B)$ for all $A, B \in S$;
   (iii) $f(AB) = f(BA)$ for all $A, B \in S$;
   (iv) $f(A) = 0$ if and only if A is not invertible.

8. [*A bit harder*] Let $f : \mathbb{R} \to \mathbb{R}$. Translate the following statement into English:

$$\forall \varepsilon > 0 \quad \forall a \in \mathbb{R} \quad \exists \delta > 0 \quad \forall x \in \mathbb{R} \quad |x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

   Express the following statement using the symbols $\forall$, $\exists$ and $\Rightarrow$:

   > for every positive real number ε there exists a positive real number δ such that for all real numbers a and x, $|f(x) - f(a)| < \varepsilon$ whenever $|x - a| < \delta$.

   Do these statements say the same thing? Can you prove whether or not they hold for the case $f(x) = x^2$?