Geometric Group Theory

Problem Sheet 1 - Solutions

Section C

1. An infinite finitely generated group is called almost finite if all its quotients are finite groups. Show that every infinite finitely generated group has a quotient that is almost finite.

Solution Let $G =< S|R >$ be a given f.g. infinite group. We enumerate all words on S and we go through the list asking whether adding w_i to relations R results to a finite group. If it does not we add it, if not we go to the next word. If this stops we get the quotient we need. If it goes on forever we add all these countably many relators. We remark that the group we obtain is infinite. This is because finite groups are finitely presented so if it were finite we would have already all the relators in a finite stage. But now we can not add more relators so the quotient we have has the required property.

Alternatively order normal subgroups N such that G/N is infinite by inclusion. An ascending union of such subgroups has the same property since if G/N is finite it is finitely presented so we can find all its relations in a fixed subgroup in the union. So by Zorn's lemma G/N is infinite. On the other hand any quotient of it is finite.

2. Show that the following presentations are presentations of the trivial group:
i) $\langle a, b, c | aba^{-1} = b^2, bcb^{-1} = c^2, cac^{-1} = a^2 \rangle$
ii) $\langle a, b | a^n = b^{n+1}, aba = bab \rangle$
iii) $\langle a, b | ab^n a^{-1} = b^{n+1}, ba^n b^{-1} = a^{n+1} \rangle$.

Solution.

I am fairly certain the solutions below are not the shortest possible.

i) We note that we have the relations:

$$a^k ba^{-k} = b^{2^k},\ b^{-1}ab = ba,\ c^{-1}bc = cb,\ a^{-1}ca = ac$$

We have

$$cac^{-1}bca^{-1}c^{-1} = b^4,\ \ b^4cb^{-4} = c^{16}$$

so

$$(cac^{-1}bca^{-1}c^{-1})c(cac^{-1}bca^{-1}c^{-1})^{-1} = c^{16} \Rightarrow$$

$$a(c^{-1}b)ca^{-1}(cac^{-1})b^{-1}ca^{-1} = c^{16} \Rightarrow ac(bab^{-1})ca^{-1} = c^{16} \Rightarrow$$

$$ac(bab^{-1})ca^{-1} = c^{16} \Rightarrow acb^{-1}(aca^{-1}) = c^{16} \Rightarrow$$

$$acb^{-1}a^{-1}c = c^{16} \Rightarrow acb^{-1}a^{-1} = c^{15} \Rightarrow (aca^{-1})(ab^{-1}a^{-1}) = c^{15} \Rightarrow$$

$$a^{-1}cb^{-2} = c^{15} \Rightarrow a^{-1}b^{-2}(b^2cb^{-2}) = c^{15} \Rightarrow a^{-1}b^{-2} = c^{11} \Rightarrow b^2a = c^{-11} \Rightarrow ab = c^{-11}$$

Now

$$c^{11}ac^{-11} = a^{2^{11}} \Rightarrow b^{-1}a^{-1}aab = a^{2^{11}} \Rightarrow b^{-1}ab = a^{2^{11}} \Rightarrow ba = a^{2^{11}} \Rightarrow b = a^{2^{11}-1}$$

But then $aba^{-1} = b^2 \Rightarrow b = 1 \Rightarrow c = 1 \Rightarrow a = 1$.

ii) Using $aba = bab$ and induction we get $a^nba = bab^n$.

$$b^{n+1}ab = b^nbab = b^naba = ... = aba^{n+1}$$

$$aba^{n+1} = aba^na = ab^{n+2}a = a^{n+1}ba$$

So

$$b^{n+1}ab = a^{n+1}b = a^{n+1}ba \Rightarrow a = 1 \Rightarrow b^2 = b \Rightarrow b = 1$$

iii) We remark that $ba^{nk}b^{-1} = (ba^nb^{-1})^k = a^{k(n+1)}$. So $b^na^{n^n}b^{-n} = a^{(n+1)^n}$. Now

$$(ab^na^{-1})a^{n^n}(ab^na^{-1})^{-1} = ab^na^{n^n}b^{-n}a^{-1} = a^{(n+1)^n}$$

and

$$(ab^na^{-1})a^{n^n}(ab^na^{-1})^{-1} = b^{n+1}a^{n^n}b^{-(n+1)} = ba^{(n+1)^n}b^{-1}$$

Therefore

$$ba^{(n+1)^n}b^{-1} = a^{(n+1)^n} \Rightarrow ba^{n(n+1)^n}b^{-1} = a^{n(n+1)^n} \Rightarrow a^{(n+1)^{n+1}} = a^{n(n+1)^n}$$

It follows that

$$a^{(n+1)^n} = 1$$

$$ba^nb^{-1} = a^{n+1} \Rightarrow (ba^nb^{-1})^{(n+1)^{n-1}} = a^{(n+1)^n} \Rightarrow a^{n(n+1)^{n-1}} = 1$$

It follows that

$$a^{n(n+1)^{n-1}} = a^{(n+1)^n} \Rightarrow a^{(n+1)^{n-1}} = 1$$

We continue inductively and we get

$$a^{n+1} = 1 \Rightarrow a^n = 1 \Rightarrow a = 1 \Rightarrow b = 1$$