Geometric Group Theory

Problem Sheet 3

Section A

1. Show that the group $G = \mathbb{Z}^2 * \mathbb{Z}^2$ can be written as an HNN-extension over $\mathbb{Z}$. Show that G can be written non-trivially as the fundamental group of a graph of groups with 3 edges.

Solution Let's say that $G = \langle a_1, a_2, b_1, b_2 | [a_1, a_2], [b_1, b_2] \rangle$. If $H =< a_2 >$ and $\theta = id$ then G is HNN-extension over H with stable letter a_1. Similarly we can consider G as an HNN-extension over $< b_2 >$ with stable letter b_1. Finally we can see it as a graph of groups with 2 loops joined by an edge. The two loops correspond to the 2 stable letters a_1, b_1, the loops are labeled respectively by $< a_2, >, < b_2 >$, the 2 vertex groups are also $< a_2, >, < b_2 >$ and the edge joining the two loops is labeled by the trivial group.

2. Let $H = \pi_1(G, Y, a_0)$. Show that if Y is not a tree then H admits an epimorphism onto $\mathbb{Z}$.

Solution. If T is a maximal tree of Y we use the presentation of H as $\pi_1(G, Y, T)$ and we map all edge generators $e \notin T$ to 1 and all other generators to 0. This is clearly an epimorphism.

Section C

3. The fundamental group of a surface group of genus 2 has a presentation:

$$G = \langle a, b, c, d | [a, b] = [c, d] \rangle$$

where we denote by $[a, b]$ the commutator: $aba^{-1}b^{-1}$. Show that G is an amalgam of two free groups over $\mathbb{Z}$. Deduce that the word problem of G is decidable.

Solution The membership problem for $< [a, b] >$ in $F(a, b)$ is decidable and similarly for $[c, d] >$ in $F(c, d)$. We remark that

$$G = F(a, b) *_< [a, b] = [c, d] > F(c, d)$$

so by a previous exercise the word problem of G is decidable.

4. Show that the group

$$G = \langle x, y | xy^2x^{-1} = y^3 \rangle$$

is not Hopf. (*hint:* consider the homomorphism $x \to x$, $y \to y^2$ and find an element in the kernel).

Solution. Let $\varphi(x) = x$, $\varphi(y) = y^2$. Then $\varphi(xy^2x^{-1}) = xy^4x^{-1} = y^6 = \varphi(y^3)$ so φ can be extended to a homomorphism. Now $\varphi(xyx^{-1}) = xy^2x^{-1} = y^3$, so $y \in im\,\varphi$ and φ is onto.

By the presentation of G we see that G is an HNN-extension of the form $< y > *_{<y^2>}$ with stable letter x and $\theta : y^2 \to y^3$.

Note that $y^{-4}xyx^{-1}yxyx^{-1}$ is a reduced word so it is $\neq 1$ and it lies in $ker\,\varphi$, so G is not Hopf.

5. Show that the product of two free groups of rank 2, $F_2 \times F_2$ can not be written as a non trivial amalgam over $\mathbb{Z}$.

Solution. If $G = F_2 \times F_2$ is an amalgam over $\mathbb{Z}$ then G acts on a tree T with all edge stabilizers isomorphic to $\mathbb{Z}$. Moreover there is an element of G that does not fix any vertex of T (indeed eg any element with normal form of length 2 has this property). So let's say that $a = (g, h) \in G$ acts by translations on a line $L \subset T$. We claim that an element of the form $(1, g)$ or $(h, 1)$ also acts on L by translations. This is clear if $g = 1$ or $h = 1$. Otherwise we remark that both $x = (g, 1)$, $y = (1, h)$ commute with a. So $xax^{-1}(x(L)) = xa(L) = x(L)$. Also $xax^{-1}(x(L)) = a(x(L))$. Hence $x(L)$ is a line invariant by a. However L is the only line invariant by a, so $x(L) = L$. Similarly $y(L) = L$. Now at least one of x, y does not fix L, so it acts on L by translations. Let's say $(g, 1)$ acts on L by translations.

We remark now that $x = (g, 1)$ commutes with $(1, c), (1, d)$ where c, d are generators of F_2. So both $c_1 = (1, c), d_1 = (1, d)$ act on L by translations. If n, k, m are the translation lengths of x, c_1, d_1 respectively then $x^k c_1^{-n}, x^m d_1^{-n}$ both fix L and generate a free group.

So an edge stabilizer contains a free group of rank 2, which is a contradiction.

6. Let $H = \pi_1(G, Y, a_0)$. If $v \in V$ consider the normal subgroup of G_v, $N = \langle\langle \alpha_e(G_e) : e \in E(Y) \text{ with } t(e) = v \rangle\rangle$. Show that there is an epimorphism $f : H \to G_v/N$.

Solution. Let T be a maximal tree of Y. We map G_v to G_v/N by projection and all the other generators of H to the identity. By definition all relators of $\pi_1(G, Y, T)$ are satisfied since if $t(e) = v$ $e\alpha_e(g)e^{-1} = ee^{-1} = 1 = \alpha_{\bar{e}}(g)$.

7. Show that every finitely generated subgroup of $F_n *_{<c>} F_n$ (F_n free of rank n) is finitely presented.

Solution. Let H be a finitely generated subgroup of $F_n *_{<c>} F_n$. Since $F_n *_{<c>} F_n$ acts on a tree T with stabilizers of vertices isomorphic to F_n and edge stabilizers conjugates of $< c >$ we have that H acts on T as well and $X = T/H$ is a graph of groups with free vertex groups and cyclic edge groups. Since H is finitely generated all generators of H can be represented as reduced words with underlying paths contained in a finite subgraph Y of T/H. If a_0 is a vertex of Y we claim that $H = \pi_1(A, Y, a_0)$. Indeed $H = \pi_1(A, X, a_0)$ and there are no S-reduced paths in $H = \pi_1(A, X, a_0)$ which are not contained in Y. This is because every element of H is represented by a unique reduced path and all elements of H are represented by S-reduced paths contained in Y. Finally remark that vertex groups are finitely generated free groups, since H is finitely generated. Indeed by the previous exercise if a vertex group is not finitely generated we have an epimorphism from H to a free group of infinite rank, which is impossible. Now we see that H can be obtained by finitely many amalgmations or HNN extensions over cyclic groups starting from a free group. These operations preserve finite presentation. So H is finitely presented.