Professor Arguz **B3.2 Geometry of Surfaces** **MT 2025**

Problem Sheet 2 Parts A and C solutions

Part A.

1.

a) State the definition of the Euler characteristic of a surface.

b) Using the triangulation of a sphere which looks like the regular octahedron, compute the Euler characteristic of the sphere. How many vertices, edges, and faces are there?

c) Let M be a surface obtained as follows: Take the top half of a sphere which has a circle (the equator) as its boundary. Then, identify every point on the equator with its antipodal, the opposite point on the equator. Note that the resulting surface M does not have any boundary. Using the triangulation of the sphere given in part b), compute the Euler characteristic of M.

Solution: **a)** Let S be a surface with a triangulation having V vertices, E edges and F faces. The *Euler characteristic* of S is

$$\chi(S) = V - E + F.$$

This number is independent of the chosen triangulation.

b) Consider the triangulation of the sphere given by a regular octahedron. It has

$$V = 6, \qquad E = 12, \qquad F = 8.$$

Therefore,

$$\chi(S^2) = 6 - 12 + 8 = 2.$$

c) Let M be the surface obtained by taking the top half of the sphere and identifying antipodal points on the equator.

Using the octahedral triangulation, place one vertex at the north pole, one at the south pole, and four vertices on the equator. The top hemisphere contains

$$V_{\text{top}} = 5, \qquad E_{\text{top}} = 8, \qquad F_{\text{top}} = 4.$$

Under the antipodal identification on the equator, the four equatorial vertices are identified in pairs, so they give two vertices. Hence the quotient surface M has

$$V = 3.$$

The four edges from the north pole to the equator are identified in pairs, giving two edges, and the four equatorial edges are also identified in pairs, giving two edges. Thus

$$E = 4.$$

The four triangular faces are identified in pairs, giving

$$F = 2.$$

Therefore,

$$\chi(M) = V - E + F = 3 - 4 + 2 = 1.$$

Thus the surface M has Euler characteristic 1 (indeed $M \cong \mathbb{RP}^2$).

Part C.

1. Let X be the compact, connected surface obtained from a planar model given by a single polygon in the plane whose boundary word (read cyclically with orientations) is

$$abca^{-1}d^{-1}ec^{-1}b^{-1}e^{-1}d.$$

That is, the sides of the polygon are identified in pairs according to this word.

1. Compute the Euler characteristic $\chi(X)$.
2. Determine whether X is orientable.
3. Identify X up to homeomorphism in the classification of compact surfaces.

Solution: The boundary word

$$abca^{-1}d^{-1}ec^{-1}b^{-1}e^{-1}d$$

describes a single 10–gon whose edges are identified in pairs.

There is one face, so $F = 1$. Each letter a, b, c, d, e appears exactly twice, hence the 10 edges are paired into 5 edge–classes, so $E = 5$.

Tracing the edge identifications shows that the 10 vertices are identified into 4 equivalence classes. Therefore

$$\chi(X) = V - E + F = 4 - 5 + 1 = 0.$$

Each edge appears once with exponent $+1$ and once with exponent -1, so all identifications reverse orientation along edges. Hence the quotient surface is orientable.

Since X is a closed, orientable surface with Euler characteristic 0, the classification theorem gives

$$\chi = 2 - 2g \;\Rightarrow\; g = 1.$$

Thus X is homeomorphic to the torus T^2.