Professor Arguz **B3.2 Geometry of Surfaces** **MT 2025**

Problem Sheet 3 Parts A and C solutions

Part A.

1.

Let S be the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \qquad a, b, c > 0,$$

and let K denote its Gaussian curvature. Show that

$$\iint_S K\, dA = 4\pi.$$

Would the same result hold for

$$S' = \{(x, y, z) \in \mathbb{R}^3 : \ x^2 + y^2 + z^4 = 1\}\,?$$

Solution: The ellipsoid is a compact, oriented surface homeomorphic to the 2-sphere, so its Euler characteristic is $\chi(S) = 2$. The (global) Gauss–Bonnet Theorem states

$$\iint_S K\, dA = 2\pi\chi(S) = 2\pi \cdot 2 = 4\pi.$$

The second surface $S' = \{x^2 + y^2 + z^4 = 1\}$ is also homeomorphic to the sphere. Therefore the same argument applies.

2. Show that there is no point on a smooth surface with Gaussian curvature 30 and mean curvature 1.

Solution: Assume that at a point on a smooth surface the Gaussian curvature and mean curvature (with the convention $H = \kappa_1 + \kappa_2$) satisfy

$$K = 30, \qquad H = 1.$$

Let κ_1, κ_2 denote the principal curvatures at that point. Then

$$\kappa_1 + \kappa_2 = 1, \qquad \kappa_1\kappa_2 = 30.$$

Hence κ_1 and κ_2 are the roots of the quadratic polynomial

$$t^2 - (\kappa_1 + \kappa_2)t + \kappa_1\kappa_2 = t^2 - t + 30.$$

The discriminant of this polynomial is

$$\Delta = (-1)^2 - 4 \cdot 1 \cdot 30 = 1 - 120 = -119 < 0.$$

Therefore the roots are not real. This is impossible, since the principal curvatures are eigenvalues of the shape operator, which is self-adjoint and hence has real eigenvalues. Consequently, there is no point on a smooth surface with Gaussian curvature $K = 30$ and mean curvature $H = 1$.

Part C.

1.

A parametrization of the unbounded cylinder S of radius a is

$$\mathbf{r}(\theta, z) = (a\cos\theta, a\sin\theta, z), \quad \theta \in [0, 2\pi),\ z \in \mathbb{R}.$$

(i) Find the first and second fundamental forms I, II for S.

(ii) Find the principal curvatures of S at the origin, using the shape operator $S = I^{-1}II$ (this convention is slightly different from that used in class).

(iii) Find the Gaussian and mean curvatures at the origin.

Solution

$$\mathbf{r}_\theta = (-a\sin\theta,\ a\cos\theta,\ 0), \qquad \mathbf{r}_z = (0, 0, 1).$$

Thus

$$E = \mathbf{r}_\theta \cdot \mathbf{r}_\theta = a^2, \quad F = \mathbf{r}_\theta \cdot \mathbf{r}_z = 0, \quad G = \mathbf{r}_z \cdot \mathbf{r}_z = 1,$$

so the matrix of the first fundamental form in the basis $\{\frac{\partial}{\partial\theta} = \partial_\theta, \frac{\partial}{\partial z} = \partial_z\}$ is

$$I = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$\mathbf{r}_\theta \times \mathbf{r}_z = (a\cos\theta,\ a\sin\theta,\ 0),$$

so the outward unit normal is

$$\mathbf{n} = \frac{1}{a}(a\cos\theta,\ a\sin\theta,\ 0) = (\cos\theta, \sin\theta, 0).$$

$$\mathbf{r}_{\theta\theta} = (-a\cos\theta,\ -a\sin\theta,\ 0) = -a(\cos\theta, \sin\theta, 0), \qquad \mathbf{r}_{\theta z} = \mathbf{r}_{zz} = (0,0,0).$$

Hence

$$L = \mathbf{r}_{\theta\theta}\cdot\mathbf{n} = -a, \quad M = \mathbf{r}_{\theta z}\cdot\mathbf{n} = 0, \quad N = \mathbf{r}_{zz}\cdot\mathbf{n} = 0,$$

and the second fundamental form matrix is

$$II = \begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix}.$$

The shape operator (Weingarten map), using the convention $S = I^{-1}II$, is given by

$$I^{-1} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S = I^{-1}II = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{a} & 0 \\ 0 & 0 \end{pmatrix}.$$

The eigenvalues of S are

$$\kappa_1 = -\frac{1}{a}, \qquad \kappa_2 = 0.$$

Consequently the Gaussian and mean curvatures are

$$K = \kappa_1\kappa_2 = 0, \qquad H = \tfrac{1}{2}(\kappa_1 + \kappa_2) = -\frac{1}{2a}.$$