Professor Arguz **B3.2 Geometry of Surfaces** **MT 2025**

Problem Sheet 4 Parts A and C solutions

Part A.

1. For each statement below, prove if the statement is true. Otherwise, disprove by giving a counterexample.

(i) If two surfaces S_1 and S_2 have the same first fundamental forms, then their Gaussian curvatures are the same.

(ii) If S is a minimal surface with Gaussian curvature zero, then S must be part of a plane.

Solution

(i) is True by Gauss' Theorema Egregium.

(ii) is True since if S is minimal, then the mean curvature $H = \kappa_1 + \kappa_2 = 0$ and if the Gaussian curvatire $\kappa = \kappa_1 \cdot \kappa_2 = 0$, then we must have $\kappa_1 = \kappa_2 = 0$, which imposes that S must be part of a plane.

2.

Let C be the cylinder

$$C = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\},$$

and let

$$\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ at \end{pmatrix}, \qquad a > 0,$$

be a helix on C. Compute the geodesic curvature of γ.

Solution:

$$\mathbf{r}(t) = (\cos t,\ \sin t,\ at), \qquad a > 0\,.$$

The speed is

$$\|\mathbf{r}'(t)\| = \sqrt{1 + a^2}\,.$$

The unit tangent is

$$\underline{t} = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{1 + a^2}}(-\sin t,\ \cos t,\ a).$$

Differentiate $\underline{t}$ with respect to the arclength parameter s:

$$\underline{t}' = \frac{d\underline{t}}{ds} = \frac{1}{1+a^2}(-\cos t,\ -\sin t,\ 0).$$

For the cylinder $x^2 + y^2 = 1$ the outward unit normal at $\gamma(t)$ is

$$\underline{n} = (\cos t,\ \sin t,\ 0).$$

Compute the cross product $\underline{n} \wedge \underline{t} = \underline{n} \times \underline{t}$:

$$\underline{n} \wedge \underline{t} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos t & \sin t & 0 \\ -\sin t/\sqrt{1+a^2} & \cos t/\sqrt{1+a^2} & a/\sqrt{1+a^2} \end{vmatrix} = \Big(\frac{a\sin t}{\sqrt{1+a^2}},\ -\frac{a\cos t}{\sqrt{1+a^2}},\ 1\Big).$$

Finally evaluate the geodesic curvature using the definition:

$$K_g = \underline{t}' \cdot (\underline{n} \wedge \underline{t}) = \frac{1}{1+a^2}(-\cos t,\ -\sin t,\ 0) \cdot \Big(\frac{a\sin t}{\sqrt{1+a^2}},\ -\frac{a\cos t}{\sqrt{1+a^2}},\ 1\Big).$$

Hence,

$$\boxed{K_g = 0.}$$

Thus, the helix is a geodesic of the cylinder.

Part C.

1. A surface of revolution is generated by revolving a plane curve

$$\gamma(u) = (r(u), z(u)), \qquad r(u) \geq 0,$$

about the z-axis. A standard parametrization of the surface is

$$\mathbf{r}(u,\theta) = \big(r(u)\cos\theta,\ r(u)\sin\theta,\ z(u)\big), \qquad \theta \in [0, 2\pi).$$

A parallel of the surface is a curve obtained by fixing $u = u_0$ and varying θ:

$$\theta \longmapsto \mathbf{r}(u_0,\theta) = \big(r(u_0)\cos\theta,\ r(u_0)\sin\theta,\ z(u_0)\big).$$

Show that if all parallels of a surface of revolution are S are geodesics, then S must be a circular cylinder.

Solution: Let S be a surface of revolution. Without loss of generality we may assume S is obtained by rotating $\alpha(u) = (f(u), g(u))$ around the z-axis. So, S is parametrized by $r(u,v) = (f(u)\cos u, f(u)\sin v, g(u))$. Fix $u = u_0$. The parallel is

$$\alpha(\theta) = \mathbf{r}(u_0, \theta).$$

Recall a curve on a surface is a geodesic iff its acceleration is normal to the surface. We compute

$$\alpha''(\theta) = \bigl(-f(u_0)\cos\theta,\ -f(u_0)\sin\theta,\ 0\bigr).$$

The surface normal is proportional to

$$\mathbf{r}_u \times \mathbf{r}_\theta = \bigl(-fg'\cos\theta,\ -fg'\sin\theta,\ ff'\bigr).$$

If $f'(u_0) = 0$, then along the parallel

$$\mathbf{r}_u \times \mathbf{r}_\theta = \bigl(-fg'\cos\theta,\ -fg'\sin\theta,\ 0\bigr),$$

which is parallel to $\alpha''(\theta)$. Hence the acceleration of the parallel is normal to the surface, and the parallel is a geodesic.