Additional (Non-examinable) Material for Electromagnetism

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$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$ (Gauss' law)	$\nabla \cdot \mathbf{B} = 0$ (no magnetic monopoles)
$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ (Faraday's law)	$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} \right)$ (Ampère-Maxwell law)

About these notes

This document collects additional material for the B7.2 Electromagnetism course, which is a third year course in the mathematics syllabus at the University of Oxford. Nothing from this document is examinable, either because the material is slightly off-syllabus, or because it is more difficult. We refer to sections and equations from the lecture notes as (L. . .). Please send any questions/corrections/comments to mark.mezei@maths.ox.ac.uk.

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2.5 Complex analytic methods

One can also solve certain boundary value problems in electrostatics using *complex analysis*. For example, for problems that have translational symmetry in the z-axis direction, so that $\phi = \phi(x, y)$ depends only on the x and y coordinates, the Laplace equation in three dimensions (L2.62) effectively reduces to the Laplace equation in *two dimensions*:

$$0 \; = \; \nabla^2 \phi \; = \; \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \; . \tag{2.1}$$

Introduce the complex coordinate

$$\mathfrak{z} \; \equiv \; x + \mathrm{i} y \; , \tag{2.2}$$

thus identifying $\mathbb{R}^2 \cong \mathbb{C}$ with the complex plane. Recall that a function f is said to be *holomorphic* in a domain $U \subseteq \mathbb{C}$ if it is complex differentiable at every point of U. We write the real and imaginary parts of f as $u(x, y) \equiv \mathrm{Re}\, f(\mathfrak{z})$, $v(x, y) \equiv \mathrm{Im}\, f(\mathfrak{z})$, viewed as functions on $\mathbb{R}^2 \cong \mathbb{C}$. Then a result in complex analysis shows that both u and v are *harmonic functions*, *i.e.* $\phi = u$ and $\phi = v$ both satisfy (2.1).

Example Taking $f(\mathfrak{z}) = \mathfrak{z}^2$, we have $u + \mathrm{i}v \; = \; (x + \mathrm{i}y)^2 \; = \; x^2 - y^2 + \mathrm{i}(2xy)$, and hence both $\phi = u = x^2 - y^2$ and $\phi = v = 2xy$ are harmonic.

This example is of course particularly simple, but it immediately solves an interesting electrostatics problem: notice that the equipotentials for $\phi(x, y) \equiv xy$ are *rectangular hyperbolae*, $xy = \text{constant}$. It follows that ϕ solves the Dirichlet problem for the first quadrant $R \equiv \{x, y \geq 0\}$, bounded by the positive x-axis and positive y-axis, with $\phi\,|_{\partial R} \; = \; 0$. Physically, this then models the electrostatic potential outside the right-angled corner of a conductor! Further examples, that also have interesting physical applications, may be found in the Feynman lectures.

3 Macroscopic media

3.4 More on magnetic dipoles

As for dielectrics, we can make all of this more quantitative by studying magnetic dipoles in more detail. An electric dipole can be constructed from two point charges $\pm q$, in a limit where the charges coalesce. There is a similar construction for a magnetic dipole, although it is a little more fiddly as magnetic fields are generated from *currents*, not point charges. We start from the general formula (L3.19) for the vector potential:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_R \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' \ . \tag{3.1}$$

Here $\mathbf{J}\,\delta V' = \rho\,\mathbf{v}\,\delta V' = q\,\mathbf{v}$, where q is the charge in the small volume $\delta V'$, centred at position $\mathbf{r}'$. On the other hand, precisely as in our derivation of the integral Biot-Savart law formula (L3.11), we may identify $I\,\delta\mathbf{r}' = q\,\mathbf{v}$ for a current I flowing through a loop C, with segment $\delta\mathbf{r}'$. Thus such a loop of current generates a magnetic vector potential

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_C \frac{\mathrm{d}\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \ . \tag{3.2}$$

To construct a point magnetic dipole at the origin, we take C to be a small circle, where we choose our coordinate axes so that this lies in the (x, y)-plane, with centre at the origin. Then $\mathbf{r}' = (a\cos\varphi', a\sin\varphi', 0)$ parametrizes C, with $a > 0$ the radius of this circle, and the Taylor expansion (L3.37) gives

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_C \mathrm{d}\mathbf{r}'\left[\frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + O(|\mathbf{r}'|^2)\right] \ . \tag{3.3}$$

We will eventually take $|\mathbf{r}'| = a \to 0$. The first term in the expansion (3.3) is zero, as the fundamental theorem of calculus gives $\int_C \mathrm{d}\mathbf{r}' \equiv \int (\mathrm{d}\mathbf{r}'/\mathrm{d}\varphi')\,\mathrm{d}\varphi' = \mathbf{0}$, since the circle C is a closed loop. Thus with $\mathrm{d}\mathbf{r}' = (-a\sin\varphi', a\cos\varphi', 0)\,\mathrm{d}\varphi'$, $\mathbf{r} = (x, y, z)$ we may evaluate (3.3) explicitly as

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0 I}{4\pi r^3}\int_0^{2\pi} (-a\sin\varphi'\,,\, a\cos\varphi'\,,\, 0\,)\left[x\,a\cos\varphi' + y\,a\sin\varphi' + O(a^2)\right]\,\mathrm{d}\varphi' \\ &= \frac{\mu_0 I}{4\pi r^3}\left[\pi a^2\,\mathbf{e}_3\times\mathbf{r} + O(a^3)\right] \ .\end{aligned} \tag{3.4}$$

Comparing to the magnetic dipole vector potential (L3.50), we thus *define*

$$\mathbf{m} \equiv I\pi a^2\,\mathbf{e} = I\cdot\mathrm{area}(C)\,\mathbf{e} \ , \tag{3.5}$$

where $\mathrm{area}(C) = \pi a^2$ is the area enclosed by C, and $\mathbf{e}$ is a unit vector perpendicular to this surface, which here is $\mathbf{e} = \mathbf{e}_3$ because of how we aligned our coordinate axes. Although we have only derived (3.5) for a circular loop, it holds for an arbitrary loop of current confined to a plane. By analogy with the electric dipole, we then take $I \to \infty$ and $a \to 0$, holding $\mathbf{m}$ fixed. In this limit the $O(a^3)$ terms in (3.4) do not contribute, and the vector potential (3.4) is *exactly* the dipole vector potential (L3.50).

If we place such a magnetic dipole in an external magnetic field $\mathbf{B}$, what force does it experience? The dipole is generated by the circular current I in C, and it is the magnetic component $q\,\mathbf{v}\times\mathbf{B}$ of the Lorentz force (L3.8) that acts on the moving charges in this current. Recalling that $q\,\mathbf{v} = I\,\delta\mathbf{r}'$ for an element of current, we sum these forces to obtain

$$\begin{aligned}
\mathbf{F} &= \int_C I\,\mathrm{d}\mathbf{r}'\times\mathbf{B}(\mathbf{r}') \\
&= I\int_0^{2\pi}(-a\sin\varphi'\,,\,a\cos\varphi'\,,\,0\,)\times\left[\mathbf{B}(\mathbf{0})+\partial_x\mathbf{B}(\mathbf{0})\,a\cos\varphi'+\partial_y\mathbf{B}(\mathbf{0})\,a\sin\varphi'+O(a^2)\right]\,\mathrm{d}\varphi' \\
&= I\left\{\pi a^2\left[\mathbf{e}_2\times\partial_x\mathbf{B}(\mathbf{0})-\mathbf{e}_1\times\partial_y\mathbf{B}(\mathbf{0})\right]+O(a^3)\right\} \\
&= I\left\{\pi a^2\left[\partial_x B_3(\mathbf{0})\,\mathbf{e}_1+\partial_y B_3(\mathbf{0})\,\mathbf{e}_2-(\partial_x B_1(\mathbf{0})+\partial_y B_2(\mathbf{0}))\,\mathbf{e}_3\right]+O(a^3)\right\} \\
&= I\left\{\pi a^2\left[\nabla\left(B_3(\mathbf{0})\right)-(\nabla\cdot\mathbf{B}(\mathbf{0}))\,\mathbf{e}_3\right]+O(a^3)\right\}\ . \qquad (3.6)
\end{aligned}$$

Here in the second line we have Taylor expanded $\mathbf{B}$ about the origin, where the dipole is, and have then proceeded to evaluate the integral and cross products explicitly. Taking the point magnetic dipole limit, where $\mathbf{m} = I\pi a^2\,\mathbf{e}_3$, then gives

$$\mathbf{F} \;=\; \nabla(\mathbf{m}\cdot\mathbf{B}) - (\nabla\cdot\mathbf{B})\,\mathbf{m} \;=\; \nabla(\mathbf{m}\cdot\mathbf{B})\ , \qquad (3.7)$$

where we have used the Maxwell equation $\nabla\cdot\mathbf{B} = 0$. This force is conservative, with potential

$$\mathbf{F} \;=\; -\nabla V_{\mathrm{dipole}}\ , \qquad V_{\mathrm{dipole}} \;\equiv\; -\mathbf{m}\cdot\mathbf{B}\ . \qquad (3.8)$$

Remarkably, this is exactly the same as for the force on an electric dipole in (L4.2), where we replace electric dipole moment $\mathbf{p}$ by magnetic dipole moment $\mathbf{m}$, and electric field $\mathbf{E}$ by magnetic field $\mathbf{B}$!

The torque about the origin is

$$\boldsymbol{\tau} \;=\; \int_C \mathbf{r}'\times\left[\,I\,\mathrm{d}\mathbf{r}'\times\mathbf{B}(\mathbf{r}')\,\right] \;=\; I\int_C \mathrm{d}\mathbf{r}'\left[\,\mathbf{r}'\cdot\mathbf{B}(\mathbf{r}')\,\right]\ , \qquad (3.9)$$

where we have used the vector triple product identity (LA.6), together with the fact that $\mathbf{r}'$ is orthogonal to $\mathrm{d}\mathbf{r}'$ for the circle C. We may similarly compute this to obtain

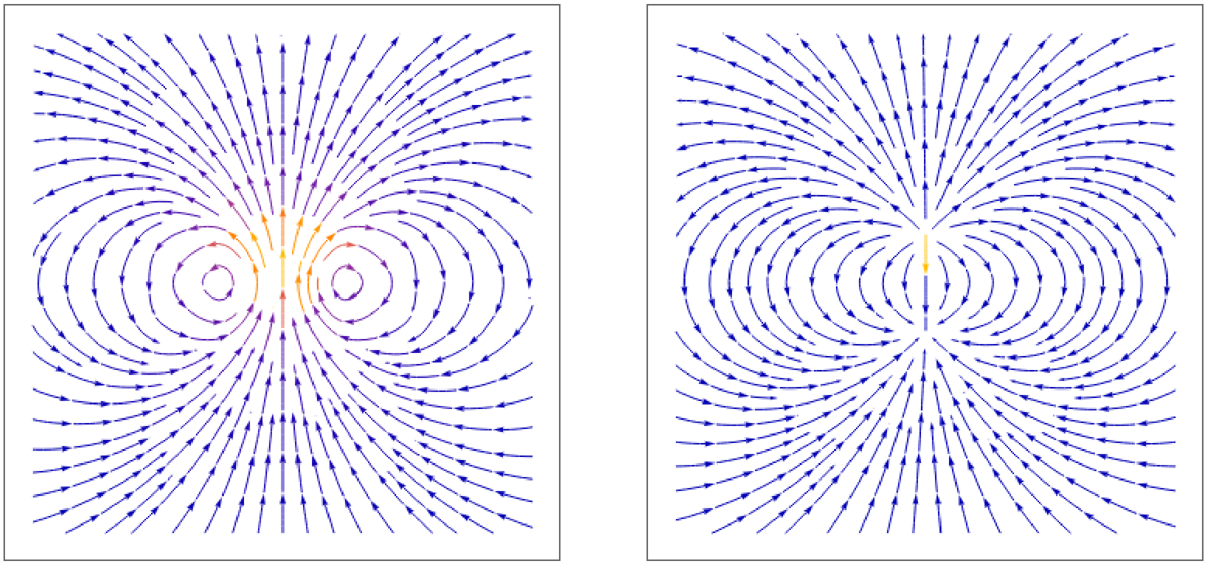
$$\begin{aligned}
\boldsymbol{\tau} &= I\int_0^{2\pi}(-a\sin\varphi'\,,\,a\cos\varphi'\,,\,0\,)\left[B_1(\mathbf{0})\,a\cos\varphi'+B_2(\mathbf{0})\,a\sin\varphi'+O(a^2)\right]\,\mathrm{d}\varphi' \\
&= I\left[\pi a^2(-B_2(\mathbf{0})\,,\,B_1(\mathbf{0})\,,\,0\,)+O(a^3)\right]\;\rightarrow\;\mathbf{m}\times\mathbf{B}\ . \qquad (3.10)
\end{aligned}$$

Again, (3.10) is the same as the torque (L4.4) on an electric dipole, but replacing $\mathbf{p}\to\mathbf{m}$, $\mathbf{E}\to\mathbf{B}$.

Magnetic dipoles in an external magnetic field thus behave in exactly the same way as electric dipoles behave in an external electric field. In particular, magnetic dipoles will tend to align everywhere with $\mathbf{B}$. This explains why iron filings align in an external magnetic field: an iron filing behaves as a magnetic dipole, due the alignment of electron spins within it. But actually you might have noticed that even more is true: the electric and magnetic fields produced by point dipoles at the origin are respectively (for $r \neq 0$)

$$\mathbf{E}_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\left[-\frac{\mathbf{p}}{r^3} + \frac{3(\mathbf{p}\cdot\mathbf{r})\,\mathbf{r}}{r^5}\right] , \qquad \mathbf{B}_{\text{dipole}} = \frac{\mu_0}{4\pi}\left[-\frac{\mathbf{m}}{r^3} + \frac{3(\mathbf{m}\cdot\mathbf{r})\,\mathbf{r}}{r^5}\right] . \tag{3.11}$$

They take exactly the same form!



(a) The magnetic field generated by a small circular current loop lying in a horizonal plane perpendicular to the page.

(b) The electric field generated by nearby point charges lying on a vertical axis, with the positive charge above the negative charge.

Figure 1: Comparing dipoles: the field lines look identical far from the centre, as in (3.11), but notice they point *upwards* in the middle of Figure 1a, and *downwards* in the middle of Figure 1b.

This might lead you to suspect there is more than just an analogy going on here: does this mean that a point magnetic dipole can *also* be constructed from two *point magnetic charges*? To some extent the answer is yes, at least mathematically, but *conceptually* this is wrong: as far as we know, point magnetic charges don't exist. Nevertheless, some textbooks introduce point magnetic charges for precisely this purpose, and indeed many physicists will then use this model when thinking about the behaviour of magnetic fields. For example, does the north pole of one magnetic dipole attract or repel the north pole of another magnetic dipole? The answer can be determined using (3.8) and (3.10), and knowing the form of the dipole magnetic field in Figure L17; but viewing the north poles as positive point magnetic charges makes it immediately clear they repel, which is correct! More fundamentally though, in (3.11) have taken an idealized point dipole limit: the field lines near the "core" of a *finite* sized current loop and pair of nearby point charges $\pm q$ actually point in opposite directions – see Figure 1.

The effective magnetostatic Maxwell equations in a material can be derived in a precisely analogous way to those for electrostatics in a dielectric medium. A large number of magnetic dipole moments $\mathbf{m}_i$ at positions $\mathbf{r}_i$ will generate a vector potential

$$\mathbf{A}(\mathbf{r})_{\text{dipoles}} = \frac{\mu_0}{4\pi}\sum_{i=1}^{N}\frac{\mathbf{m}_i\times(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3} \to \frac{\mu_0}{4\pi}\int_R \frac{\mathbf{M}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,\mathrm{d}V' \,, \tag{3.12}$$

with continuum limit precisely as in (L4.6), so that $\mathbf{M}(\mathbf{r}')\,\delta V'$ is the magnetic dipole moment in a small volume $\delta V'$, centred at position $\mathbf{r}'$.

Definition The vector field $\mathbf{M}$ is called the *magnetization density*.

A similar computation to (L4.7) then gives

$$\begin{aligned}\mathbf{A}(\mathbf{r})_{\text{dipoles}} &= \frac{\mu_0}{4\pi}\int_R \mathbf{M}(\mathbf{r}')\times\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)\mathrm{d}V' = \frac{\mu_0}{4\pi}\int_R\left[\frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \nabla'\times\left(\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right)\right]\mathrm{d}V' \\ &= \frac{\mu_0}{4\pi}\int_R \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' + \frac{\mu_0}{4\pi}\int_{\partial R}\frac{\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\times\mathrm{d}\mathbf{S}' \\ &= \frac{\mu_0}{4\pi}\int_R \frac{\nabla'\times\mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,\mathrm{d}V' \,. \end{aligned} \tag{3.13}$$

Here in the second line we have used a corollary of the divergence theorem to write the volume integral of a curl as a boundary integral of a cross product, and in the last step we have assumed that $\mathbf{M}$ is zero on the boundary of R, the region containing the magnetic dipoles. Comparing (3.13) to (L3.19) we may *define*

$$\mathbf{J}_M \equiv \nabla\times\mathbf{M} \,. \tag{3.14}$$

The subscript M here denotes these are *effective magnetizing currents*, that generate the vector potential (3.13) due to magnetic dipoles in the material.

We may then divide the electric current in Ampère's law (L3.28) into a *free current density* $\mathbf{J}_{\text{free}}$ and the magnetizing current $\mathbf{J}_M$ in (3.14). Thus

$$\nabla\times\mathbf{B} = \mu_0\,\mathbf{J} = \mu_0\,(\mathbf{J}_{\text{free}} + \mathbf{J}_M) = \mu_0\,\mathbf{J}_{\text{free}} + \mu_0\,\nabla\times\mathbf{M} \,. \tag{3.15}$$

On the other hand, the magnetization density $\mathbf{M}$ will align everywhere with the magnetic field $\mathbf{B}$, so that the cross product $\mathbf{M}(\mathbf{r})\times\mathbf{B}(\mathbf{r}) = \mathbf{0}$, and there is no torque on the magnetic dipoles. Thus

$$\mathbf{M} \equiv \frac{\chi_m}{\mu}\mathbf{B} \equiv \left(\frac{1}{\mu_0} - \frac{1}{\mu}\right)\mathbf{B} \,. \tag{3.16}$$

Definition μ is called the *permeability*, with $\chi_m \equiv (\mu/\mu_0 - 1)$ the *magnetic susceptibility*.

μ is not constant in general, but for uniform materials it is approximately constant. For the vacuum $\mu/\mu_0 = 1$, for air $\mu/\mu_0 \simeq 1.00000037$, for water $\mu/\mu_0 \simeq 0.999992$, while for iron $\mu/\mu_0 \simeq 200,000$!

Substituting (3.16) into (3.15), we have

$$\mu_0\,\mathbf{J}_{\text{free}} = \nabla\times(\mathbf{B} - \mu_0\,\mathbf{M}) = \mu_0\,\nabla\times\left(\frac{1}{\mu}\mathbf{B}\right) \,. \tag{3.17}$$

which leads to the effective Maxwell equations

$$\nabla \cdot \mathbf{B} = 0 , \qquad \nabla \times \left(\frac{1}{\mu} \mathbf{B} \right) = \mathbf{J}_{\text{free}} . \tag{3.18}$$

Notice that the magnetic field generated by $\mathbf{J}_{\text{free}}$ in a magnetic material is μ/μ_0 times the field that would be generated *without* the magnetic material present. This is *e.g.* a little larger for air, since $\chi_m > 0$ (called *paramagnetism*), but a little smaller for water, since $\chi_m < 0$ (called *diamagnetism*).

Definition The quantity $\mathbf{H} \equiv \frac{1}{\mu} \mathbf{B}$ in (3.18) is called the *magnetic field strength*.

Magnetism is more complicated than electric polarization, for a number of reasons. First, the alignment of magnetic dipoles in an external magnetic field that we have described is more specifically called *paramagnetism*. It usually results in a small positive $\chi_m > 0$, with the aligned dipoles effectively *increasing* slightly the overall magnetic field. However, there are also materials, such as water, with a small but *negative* $\chi_m < 0$. This *diamagnetism* is *not* due to the alignment of dipoles, but rather an applied magnetic field can result in a change in electric currents in the medium (at the atomic scale, by changing electron orbits), which in turn generates a magnetic field in the *opposite* direction. These two effects in general compete, and which is dominant depends on precise atomic/molecular structure. Finally, *ferromagnetic* materials, such as iron, become magnetized under even a small applied magnetic field, and moreover then *remain* magnetized. Here the alignment of (spin) magnetic dipoles in one region influences the alignment in neighbouring regions – our discussion ignored dipole-dipole interactions, which in ferromagnetic materials are important.

5 Electrodynamics and Maxwell's equations

5.6 Time-dependent Green's function

5.6.2 Radiation from an accelerated charge

To avoid confusion with derivatives, let t_r (previously t') denote the solution to the implicit equation $c(t - t_r) = R$ where $R = |\mathbf{R}|$ and $\mathbf{R} = \mathbf{r} - \mathbf{r}_0(t_r)$. With the notations $\boldsymbol{\beta} = \mathbf{v}_0(t_r)/c$ and $\mathbf{n} = \mathbf{R}(t_r)/R(t_r)$, we can write the Liénard–Wiechert potentials of a moving point charge as

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \mathbf{R} \cdot \boldsymbol{\beta}} \quad \text{and} \quad \mathbf{A}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0 c} \frac{\boldsymbol{\beta}}{R - \mathbf{R} \cdot \boldsymbol{\beta}}.$$

To compute the fields $\mathbf{E}$ and $\mathbf{B}$, we need to know the implicit dependence of $t_r = t_r(\mathbf{r}, t)$ on $\mathbf{r}$ and t. Therefore, consider the defining relation $t = t_r + R(\mathbf{r}, t_r)/c$. We regard $(\mathbf{r}, t)$ as independent variables and compute the partial derivatives of this equation to get

- Taking the t-derivative, we find

  $$1 = \frac{\partial t_r}{\partial t} + \frac{1}{c} \frac{\partial R}{\partial t_r} \frac{\partial t_r}{\partial t} = (1 - \mathbf{n} \cdot \boldsymbol{\beta}) \frac{\partial t_r}{\partial t},$$

  where in the last equality we used (L5.68).

- Taking the gradient, we find (using the chain rule)

  $$0 = \nabla t = \nabla t_r + \nabla \frac{R(\mathbf{r}, t_r)}{c} = \nabla t_r + \frac{\mathbf{n}}{c} + \frac{1}{c} \frac{\partial R}{\partial t_r} \nabla t_r = \frac{\mathbf{n}}{c} + (1 - \mathbf{n} \cdot \boldsymbol{\beta}) (\nabla t_r).$$

In summary, we have obtained that

$$\frac{\partial t_r}{\partial t} = \frac{1}{1 - \mathbf{n} \cdot \boldsymbol{\beta}} \quad \text{and} \quad \nabla t_r = -\frac{\mathbf{n}/c}{1 - \mathbf{n} \cdot \boldsymbol{\beta}}. \tag{5.1}$$

Using these, we can compute the electromagnetic fields. For $-\nabla\phi$, we need

$$\begin{aligned} -\nabla \frac{1}{R - \mathbf{R} \cdot \boldsymbol{\beta}} &= \frac{1}{R^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2} \nabla (R - \mathbf{R} \cdot \boldsymbol{\beta}) = \frac{1}{R^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2} \Big(\underbrace{\nabla R}_{\mathbf{n} + \frac{\partial R}{\partial t_r} \nabla t_r} - \underbrace{\nabla(\mathbf{R} \cdot \boldsymbol{\beta})}_{-\boldsymbol{\beta} + \frac{\partial(\mathbf{R} \cdot \boldsymbol{\beta})}{\partial t_r} \nabla t_r = -\boldsymbol{\beta} + (-c\beta^2 + \mathbf{R} \cdot \boldsymbol{\beta}') \nabla t_r} \Big) \\ &= -\frac{\boldsymbol{\beta}}{R^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2} + \frac{\mathbf{n}}{R^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \left(1 - \beta^2 + \mathbf{R} \cdot \boldsymbol{\beta}'/c\right) \end{aligned} \tag{5.2}$$

where we set $\boldsymbol{\beta}' = \mathbf{v}_0'(t_r)/c$ to the acceleration of the charge, at the retarded time divided by c, write $\beta = |\boldsymbol{\beta}| = |\mathbf{v}_0(t_r)|/c$, and to get to the last line we gathered terms proportional to the vectors $\boldsymbol{\beta}$ and $\mathbf{n}$. To compute $\partial \mathbf{A}/\partial t$, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\boldsymbol{\beta}/c}{R - \mathbf{R} \cdot \boldsymbol{\beta}} &= \left(\frac{\boldsymbol{\beta}'/c}{R - \mathbf{R} \cdot \boldsymbol{\beta}} \frac{\partial t_r}{\partial t} \right) - \frac{\boldsymbol{\beta}/c}{(R - \mathbf{R} \cdot \boldsymbol{\beta})^2} \frac{\partial}{\partial t} (R - \mathbf{R} \cdot \boldsymbol{\beta}) \\ &= \frac{\boldsymbol{\beta}'/c}{R(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2} - \frac{\boldsymbol{\beta}/c}{(R - \mathbf{R} \cdot \boldsymbol{\beta})^2} \Big(\underbrace{\frac{\partial R}{\partial t}}_{-c\mathbf{n} \cdot \boldsymbol{\beta} \frac{\partial t_r}{\partial t}} - \underbrace{\frac{\partial(\mathbf{R} \cdot \boldsymbol{\beta})}{\partial t_r}}_{-c\beta^2 + \mathbf{R} \cdot \boldsymbol{\beta}'} \frac{\partial t_r}{\partial t} \Big) \\ &= \frac{\boldsymbol{\beta}'/c}{R(1 - \mathbf{n} \cdot \boldsymbol{\beta})^2} - \frac{\boldsymbol{\beta}}{R^2(1 - \mathbf{n} \cdot \boldsymbol{\beta})^3} \left(\beta^2 - \mathbf{n} \cdot \boldsymbol{\beta} - \frac{\mathbf{R} \cdot \boldsymbol{\beta}'}{c} \right). \end{aligned} \tag{5.3}$$

Combining the two calculations above, we obtain for $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ the result

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\mathbf{n}-\boldsymbol{\beta}}{R^2(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\left(1-\beta^2+\mathbf{R}\cdot\boldsymbol{\beta}'/c\right) - \frac{\boldsymbol{\beta}'}{cR(1-\mathbf{n}\cdot\boldsymbol{\beta})^2}\right]. \tag{5.4}$$

Note that both (5.2) and (5.3) contributed to the coefficient of $\boldsymbol{\beta}$, and it turned out to equal minus that of $\mathbf{n}$. Similarly we can compute the magnetic flux density $\mathbf{B}$. The result can be stated as

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}\mathbf{n}\times\mathbf{E}. \tag{5.5}$$

Note that $\mathbf{B}$ is orthogonal to $\mathbf{E}$ and $\mathbf{n}$. Organizing the expression for $\mathbf{E}$ according to the R-scaling, we find two contributions:

$$\mathbf{E} = \frac{1}{R^2}\frac{q(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{4\pi\epsilon_0(1-\mathbf{n}\cdot\boldsymbol{\beta})^3} + \frac{1}{R}\frac{q}{4\pi\epsilon_0 c(1-\mathbf{n}\cdot\boldsymbol{\beta})^3}\underbrace{\left((\mathbf{n}-\boldsymbol{\beta})(\mathbf{n}\cdot\boldsymbol{\beta}') - \boldsymbol{\beta}'(1-\mathbf{n}\cdot\boldsymbol{\beta})\right)}_{\mathbf{n}\times((\mathbf{n}-\boldsymbol{\beta})\times\boldsymbol{\beta}')}. \tag{5.6}$$

The first contribution $\propto 1/R^2$ is Coulomb-like, but the second contribution $\propto 1/R$ falls off only linearly and thus has longer range. Note that this long-range component is only present when the charge accelerates ($\boldsymbol{\beta}' \neq 0$), hence we do not see it for a charge with constant velocity.

We conclude: *An accelerated electric charge emits electromagnetic radiation*! This is the principle behind antennas, synchrotrons, and free electron lasers.

For the Poynting vector $\boldsymbol{\mathcal{P}} = \mathbf{E}\times\mathbf{B}/\mu_0$ we find

$$\begin{aligned}\boldsymbol{\mathcal{P}} &= \mathbf{E}\times\frac{\mathbf{n}\times\mathbf{E}}{c\mu_0} = \frac{|\mathbf{E}|^2\mathbf{n} - \mathbf{E}(\mathbf{E}\cdot\mathbf{n})}{c\mu_0} \\ &= \frac{q^2}{16\pi^2\epsilon_0 c R^2}\frac{|\mathbf{n}\times\left((\mathbf{n}-\boldsymbol{\beta})\times\boldsymbol{\beta}'\right)|^2}{(1-\mathbf{n}\cdot\boldsymbol{\beta})^6}\mathbf{n} + \mathcal{O}(1/R^3).\end{aligned} \tag{5.7}$$

This energy flux density is directed along $\mathbf{n}$, that is, radiation is being *emitted* by the moving charge. Integrating $\boldsymbol{\mathcal{P}}$ over a sphere, once finds for the total emitted power

$$\frac{\mathrm{d}W}{\mathrm{d}t_r} = \left(\frac{\partial t}{\partial t_r}\right)\int\boldsymbol{\mathcal{P}}\cdot\mathrm{d}\mathbf{S} = \frac{q^2}{6\pi\epsilon_0 c}\gamma^4\left(|\boldsymbol{\beta}'|^2 + \gamma^2(\boldsymbol{\beta}\cdot\boldsymbol{\beta}')^2\right), \tag{5.8}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ and the $(\partial t/\partial t_r)$ factor accounts for the radiating charge's own time that is different from t. If one applies this formula to the model of a hydrogen atom, with the electron moving in a circular orbit (note $\boldsymbol{\beta}\cdot\boldsymbol{\beta}' = 0$)

$$\mathbf{r}_0(t) = r_0\begin{pmatrix}\cos\omega t\\ \sin\omega t\\ 0\end{pmatrix}, \qquad \boldsymbol{\beta}(t) = \frac{r_0\omega}{c}\begin{pmatrix}-\sin\omega t\\ \cos\omega t\\ 0\end{pmatrix}, \qquad \boldsymbol{\beta}'(t) = \frac{r_0\omega^2}{c}\begin{pmatrix}-\cos\omega t\\ -\sin\omega t\\ 0\end{pmatrix},$$

one finds that the entire binding energy of an electron in the ground state would be radiated away in $\approx 10^{-11}$ seconds. This contradicts the existence of hydrogen and suggests that the classical mechanical model of the atom is too simplistic. This puzzle is resolved in quantum mechanics.

7 Electromagnetism and Special Relativity

The theory of electromagnetism developed in the 19^{th} century was extraordinarily successful, unifying the previously unrelated phenomena of electricity and magnetism into a single theory. For example, it explained Faraday's law, where a time-dependent magnetic field produces an electric field, which in turn led to the development of electric motors, transformers, *etc.* As we saw in the last section, the theory also interprets visible light, along with the rest of the electromagnetic spectrum (X-rays, microwaves, radio waves, *etc*), as a wave propagating through this electromagnetic field. Maxwell identified $c = 1/\sqrt{\epsilon_0 \mu_0}$ with the speed of light in vacuum. But that also led to a problem: *speed relative to what*?

Suppose that Louisa is on a train that moves in a straight line with constant speed v relative to Franklin, who is at rest in the train station. Louisa rolls a marble along the aisle of the train, in the direction of its motion, with speed u. This means that in Louisa's inertial reference frame $\mathcal{S}'$, fixed relative to the train, the marble moves with speed u. In Frankin's inertial reference frame $\mathcal{S}$, fixed relative to the Earth's surface, what is the observed speed of the marble? It's certainly *greater* than u, due to the train's speed $v > 0$. If you asked a random person in the street, they would almost certainly say Franklin sees the marble moving with speed $u + v$. This is intuitively obvious, and wrong. It turns out it's only *approximately* true, for speeds $u, v \ll c$.

Rather than experiment with marbles, suppose that Louisa and Franklin instead measure the electrostatic force between electric charges, and the magnetostatic force between current carrying wires, and from Maxwell's equations thus measure ϵ_0, μ_0. Going back to Galileo, we have:

Postulate 1 *The laws of physics are the same in all inertial reference frames.*

By this principle, Louisa and Franklin should measure the *same* values for ϵ_0, μ_0 in their two reference frames, namely those quoted earlier in these lecture notes. But according to Maxwell they will then both observe light to be propagating at the *same* speed $c = 1/\sqrt{\epsilon_0 \mu_0}$. Light is clearly not like marbles: it's *always* moving at the same speed, no matter how your inertial reference frame is moving relative to it. If we believe that Postulate 1 (the *Principle of Relativity*) applies to electromagnetism, we are led to:

Postulate 2 *The speed of light in vacuum is the same in all inertial reference frames.*

These two postulates directly led to Einstein's 1905 theory of Special Relativity. It supersedes the Galilean view of space and time, although reduces to it in the limit of small speeds $v \ll c$.[1]

Let us examine the consequences of this a little further. Introduce time and space coordinates t, x, y, z for Franklin's reference frame $\mathcal{S}$, and t', x', y', z' for Louisa's reference frame $\mathcal{S}'$. Suppose

[1] It is worth remarking that before 1905 physicists, including Maxwell, had instead postulated that Maxwell's equations are only valid in a unique *universal rest frame*. This was supposed to be filled with something called *aether*, through which light propagated. However, a famous 1887 experiment by Michelson and Morely provided strong evidence that Postulate 2 is correct.

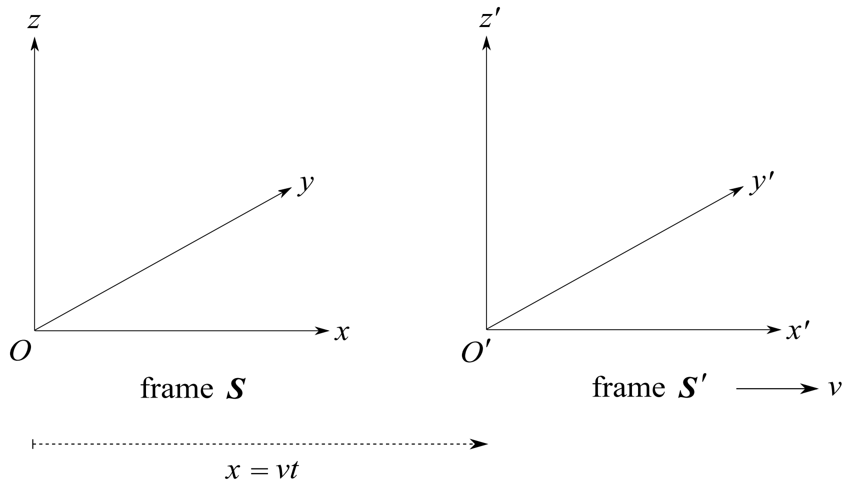


Figure 2: Reference frame $\mathcal{S}$ with coordinates t, x, y, z, and reference frame $\mathcal{S}'$ with coordinates t', x', y', z'. The origins O, O' coincide at times $t = 0 = t'$, with $\mathcal{S}'$ moving in the x-axis direction, relative to $\mathcal{S}$, with speed v. The origin O' is thus at position $x = vt$, $y = z = 0$ in the frame $\mathcal{S}$.

as above that Louisa's reference frame has speed v relative to Franklin's, moving in the x-axis direction. Suppose furthermore that their origins O, O' coincide at times $t = 0 = t'$. At this moment, a flash of light is emitted from the common origins, expanding as a spherical wave (precisely as in the retarded Green's function (L5.60)). According to Postulate 2, the speed of this wave is c in both reference frames, so in particular Franklin will see the wave obey

$$ct = |\mathbf{r}| \quad \Rightarrow \quad -c^2t^2 + x^2 + y^2 + z^2 = 0 \,, \tag{7.1}$$

in his frame $\mathcal{S}$. Here ct is the distance travelled by light in time t, while $|\mathbf{r}| \equiv \sqrt{x^2 + y^2 + z^2}$ is the distance of the point $\mathbf{r} = (x, y, z)$ from the origin O. But similarly Louisa will see the wave obey

$$ct' = |\mathbf{r}'| \quad \Rightarrow \quad -c^2t'^2 + x'^2 + y'^2 + z'^2 = 0 \,, \tag{7.2}$$

in her frame $\mathcal{S}'$.

The issue now is how these coordinates are related to each other. Galileo would say

$$\textbf{Galilean transformation}: \quad t' = t \,, \quad x' = x - vt \,, \quad y' = y \,, \quad z' = z \,. \tag{7.3}$$

This is a particular case of the more general set of Galilean transformations with

$$t' = t - t_0 \,, \qquad \mathbf{r}' = R\,\mathbf{r} - \mathbf{r}_0 - \mathbf{v}t \,. \tag{7.4}$$

Here t_0 is a constant, that is zero if the two observers synchronize their clocks; $\mathbf{r}_0$ is a constant vector, that is zero if the two observers fix a common origin at time $t = 0$; R is a 3×3 orthogonal matrix (*i.e.* a rotation and potentially also reflection of the spatial directions); and $\mathbf{v}$ is a constant velocity. The set of transformations (7.4) form a group, called the *Galilean group*. They map inertial reference frames to inertial reference frames, in particular meaning they map *uniform motion* (*i.e.* with constant velocity) in one reference from to uniform motion in the other frame.

Postulate 1 says that the law physics are the same in any inertial reference frame, and indeed Newton's laws of motion are invariant under Galilean transformations.

This Galilean view of space and time was the standard lore before 1905, but it is *not consistent* with electromagnetism. Specifically, if you substitute the Galilean transformation (7.3) into (7.2), you obtain $-c^2t^2 + (x - vt)^2 + y^2 + z^2 = 0$, which is *not* the spherical wavefront (7.1) seen in Franklin's frame. It should be clear why: according to Galileo, a ray of light travelling at speed $u = c$ in the positive x-axis direction in Louisa's frame $\mathcal{S}'$ has $x' = ut$, which in Franklin's frame $\mathcal{S}$ is $x = x' + vt = (u + v)t$, and thus has speed $u + v = c + v \neq c$. Galilean transformations are inconsistent with Postulate 2.

The transformations (7.4) are *linear* maps from $(ct, x, y, z) \in \mathbb{R}^4$ to $(ct', x', y', z') \in \mathbb{R}^4$. Here we have multiplied the time coordinate by c so that ct also has dimensions of length. Notice the maps are linear because we want to map uniform motion (which traces out straight lines in $\mathbb{R}^4$) to uniform motion. We may rewrite (7.4) as

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline -\frac{v_1}{c} & & & \\ -\frac{v_2}{c} & & R & \\ -\frac{v_3}{c} & & & \end{array}\right) \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} + \begin{pmatrix} ct_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} \equiv G \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} + \text{constant} . \tag{7.5}$$

Here the 3×3 orthogonal matrix R fills the lower right hand block of the 4×4 matrix G.

The linear transformation that maps (ct, x, y, z) to (ct', x', y', z') that is consistent with (7.1) and (7.2) is

$$\textbf{Lorentz transformation}: \quad ct' = \frac{ct - \frac{v}{c}x}{\sqrt{1 - \frac{v^2}{c^2}}} , \quad x' = \frac{x - \frac{v}{c} \cdot ct}{\sqrt{1 - \frac{v^2}{c^2}}} , \quad y' = y , \quad z' = z . \tag{7.6}$$

Specifically, one can easily verify that $-c^2t'^2 + x'^2 + y'^2 + z'^2 = -c^2t^2 + x^2 + y^2 + z^2$ under this transformation. Notice that (7.6) approximately reduces to (7.3) for speeds $v \ll c$. Since (7.6) is linear, we may write it similarly to (7.5)

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\frac{v}{c}\gamma & 0 & 0 \\ -\frac{v}{c}\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \equiv L \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} , \tag{7.7}$$

where we have introduced

$$\gamma = \gamma(v) \equiv \frac{1}{\sqrt{1 - v^2/c^2}} . \tag{7.8}$$

The Galilean transformation (7.3) has $x' = x - vt$, while the Lorentz transformation has $x' = \gamma(x - vt)$, and moreover treats the time and space directions symmetrically. Notice also that a marble moving with speed u along the positive x-axis direction in Louisa's frame $\mathcal{S}'$ has $x' = ut'$, which in terms of x and t is the equation

$$\gamma(x - vt) = u\gamma\left(t - \frac{v}{c^2}x\right) \quad \Rightarrow \quad x = \frac{(u + v)t}{1 + uv/c^2} , \tag{7.9}$$

so that the speed as seen in Franklin's frame $\mathcal{S}$ is $(u+v)/(1+uv/c^2)$. This approximately reduces to $u+v$, for $uv \ll c^2$. On the other hand, for $u = c$ the speed in the frame $\mathcal{S}$ is $(c+v)/(1+cv/c^2) = c$. Lorentz discovered these transformations by studying Maxwell's equations, realizing they were not invariant under Galilean transformations. Indeed, we noted this already in section L5.2 when motivating Faraday's law. For example, the Biot-Savart law says that moving charges generate magnetic fields, but moving relative to which reference frame? Einstein showed that the same transformations follow directly from Postulates 1 and 2, independently of Maxwell's equations.

The most striking feature of (7.6) is that the time coordinates in the two inertial frames are *not the same*, due to the factor of γ. Consider a clock at rest in Franklin's frame $\mathcal{S}$. The location of the clock on two different ticks is the same, so $\Delta x = 0$, and (7.6) gives

$$\Delta t' \ = \ \gamma \, \Delta t \ . \tag{7.10}$$

Here Δt is the time interval between ticks of the clock, as seen in Franklin's frame $\mathcal{S}$, while $\Delta t'$ is the time interval between ticks of the clock, as seen in Louisa's frame $\mathcal{S}'$. Since $\gamma > 1$ for $v \neq 0$, $\Delta t' > \Delta t$. In other words, Louisa sees the time between ticks of Franklin's clock taking *longer* than the time Δt. His clock seems to be running slow. The fact that $\Delta t' = \Delta t$ in Galileo's view of space and time was always a (tacit) *assumption*, and it is not compatible with Postulate 2.

The Lorentz transformations may be characterized mathematically as follows. We first assemble the time and space coordinates into a *four-vector* $\vec{X} \in \mathbb{R}^4$, writing $\vec{X} = (ct, x, y, z)^T$. We may then write a general Lorentz transformation, as in (7.7), as

$$\vec{X}' \ = \ L \, \vec{X} \ , \tag{7.11}$$

with L a linear map on spacetime $\mathbb{R}^4$. We then define the 4×4 diagonal matrix

$$\eta \ \equiv \ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \ , \tag{7.12}$$

(the *Minkowski metric tensor*), and note that we may write

$$-c^2t^2 + x^2 + y^2 + z^2 \ = \ \vec{X}^T \eta \, \vec{X} \ . \tag{7.13}$$

Lorentz transformations preserve this quadratic form, meaning

$$\vec{X}'^T \eta \, \vec{X}' \ \equiv \ \vec{X}^T L^T \eta \, L \, \vec{X} \ = \ \vec{X}^T \eta \, \vec{X} \qquad \text{holds for all } \vec{X} \in \mathbb{R}^4, \tag{7.14}$$

or in other words $-c^2t'^2 + x'^2 + y'^2 + z'^2 = -c^2t^2 + x^2 + y^2 + z^2$. This in turn implies

$$L^T \eta \, L \ = \ \eta \ . \tag{7.15}$$

This is the defining property of a Lorentz transformation L. One can compare to the 3×3 orthogonal transformation R, which by definition satisfies $R^T R = \mathbb{1}_{3\times 3}$, and preserves Euclidean distance so

$$\mathbf{r}'^T \, \mathbf{r}' \; \equiv \; \mathbf{r}^T R^T R \, \mathbf{r} \; = \; \mathbf{r}^T \, \mathbf{r} \qquad \text{holds for all } \mathbf{r} \in \mathbb{R}^3. \tag{7.16}$$

Indeed, rotations are contained within the Lorentz transformations as

$$L \; = \; \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array} \right) \, , \tag{7.17}$$

just as they are contained within the Galilean transformations (7.5). To summarize, Lorentz transformations preserve the *Lorentzian square distance* $-c^2t^2 + x^2 + y^2 + z^2$ in $\mathbb{R}^4$, which unlike a usual distance can be positive, negative, or zero.

After this brief detour into Special Relativity and Lorentz transformations, we return to discuss electromagnetism. We have already noted that Maxwell's equations are not invariant under Galilean transformations. Under a 3×3 rotation R, a vector such as $\mathbf{E}$ or $\mathbf{B}$ would rotate as a vector, and we have seen that Lorentz transformations naturally act on four-vectors, rather than three-vectors, but contain rotations as a special case. The correct Lorentz transformations of electromagnetism are most easily stated by first recalling that

$$\mathbf{E} \; = \; -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \, , \qquad \mathbf{B} \; = \; \nabla \times \mathbf{A} \, , \tag{7.18}$$

in terms of the potentials ϕ and $\mathbf{A}$. Recalling also that $\frac{1}{c}\,\mathbf{E}$ and $\mathbf{B}$ have the same dimensions, it is natural to put ϕ and $\mathbf{A}$ into the four-vector

$$\vec{A} \equiv \left(\frac{\phi}{c}, A_1, A_2, A_3 \right)^T \; = \; \left(\frac{\phi}{c} \, , \, \mathbf{A} \right)^T \, , \tag{7.19}$$

and similarly define the *four-current*

$$\vec{J} \equiv \left(c\,\rho \, , \, \mathbf{J} \right) \, , \tag{7.20}$$

in terms of the charge density ρ and current density $\mathbf{J}$. The four-vector $\vec{A}'$ and current $\vec{J}'$ in the frame $\mathcal{S}'$ are then simply

$$\vec{A}' \; = \; L\,\vec{A} \, , \qquad \vec{J}' \; = \; L\,\vec{J} \, , \tag{7.21}$$

where L is the Lorentz transformation from $\mathcal{S}$ to $\mathcal{S}'$. In Lorenz gauge we can write the Maxwell equations (L5.34), (L5.35) as the single four-vector equation

$$\boxed{\Box\,\vec{A} \; = \; -\mu_0\,\vec{J} \, .} \tag{7.22}$$

Moreover, notice that

$$\Box \equiv -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \sum_{i=1}^{3}\frac{\partial^2}{\partial x_i^2} = \vec{\partial}^T \eta \vec{\partial} , \tag{7.23}$$

where $\vec{\partial} \equiv (\frac{1}{c}\partial_t, \partial_x, \partial_y, \partial_z)^T$ is the gradient operator on spacetime. The d'Alembertian $\Box$ is thus the natural analogue of the Laplacian in spacetime, and is *invariant* under Lorentz transformations. It follows that Maxwell's equations (7.22) take the same form in both reference frames, with both sides transforming as a Lorentz four-vector.

Example We reconsider the example at the end of section 5.6, namely a point charge q moving with constant velocity $\mathbf{v} = v\,\mathbf{e}_1$ in the reference frame $\mathcal{S}$. We computed the potentials ϕ and $\mathbf{A}$ in equations (L5.75) and (L5.76), using the general solution to the time-dependent Maxwell equations (L5.62) and (L5.63), respectively. The point charge hence generates both an $\mathbf{E}$ and a $\mathbf{B}$ field in the frame $\mathcal{S}$. On the other hand, in the frame $\mathcal{S}'$ this charge is at *rest*. Taking this to be the origin $\mathbf{r}' = \mathbf{0}$ in $\mathcal{S}'$, the laws of statics imply that the charge generates the potentials

$$\phi'(\mathbf{r}') = \frac{q}{4\pi\epsilon_0}\frac{1}{r'} , \qquad \mathbf{A}' = \mathbf{0} , \tag{7.24}$$

in the frame $\mathcal{S}'$.

The four-vector $\vec{A}' = (\frac{\phi'}{c}, \mathbf{A}')^T$ in the frame $\mathcal{S}'$ is related the four-vector $\vec{A} = (\frac{\phi}{c}, \mathbf{A})^T$ in the frame $\mathcal{S}$ via the Lorentz transformation (7.21). Using (7.7) this reads

$$\frac{\phi'}{c} = \gamma\frac{\phi}{c} - \frac{v}{c}\gamma A_1 , \qquad A_1' = -\frac{v}{c}\gamma\frac{\phi}{c} + \gamma A_1 , \qquad A_2' = A_2 , \qquad A_3' = A_3 . \tag{7.25}$$

The Lorentz transformed postion vector is

$$\mathbf{r}' = \gamma(x - vt)\,\mathbf{e}_1 + y\,\mathbf{e}_2 + z\,\mathbf{e}_3 , \qquad r'^2 = \gamma^2(x-vt)^2 + y^2 + z^2 . \tag{7.26}$$

Since from (7.24) $\mathbf{A}' = \mathbf{0}$, one immediately solves the last three equations in (7.25) to find $\mathbf{A} = \frac{\phi}{c^2}\,\mathbf{v}$. Substituting this into the first equation in (7.25) then gives

$$\frac{\phi'}{c} = \gamma\left(1 - \frac{v^2}{c^2}\right)\frac{\phi}{c} = \frac{1}{\gamma}\frac{\phi}{c} , \tag{7.27}$$

and hence from (7.24) we deduce

$$\phi(\mathbf{r}, t) = \frac{q\gamma}{4\pi\epsilon_0}\frac{1}{\sqrt{\gamma^2(x-vt)^2 + y^2 + z^2}} , \qquad \mathbf{A} = \frac{\phi(\mathbf{r}, t)}{c^2}\,\mathbf{v} , \tag{7.28}$$

in precise agreement with (L5.75), (L5.76)! We have here derived these formulae from Coulomb's law (7.24) in the frame $\mathcal{S}'$, together with a Lorentz transformation.

The electric and magnetic fields in the frame $\mathcal{S}$ may be computed from these potentials using the usual formulae (7.18). Indeed, combining the latter with the Lorentz transformation of the potentials (7.25) leads to the transformations

$$\begin{aligned} E_1' &= E_1 , \qquad & E_2' &= \gamma\,(E_2 - vB_3) , \qquad & E_3' &= \gamma\,(E_3 + vB_2) , \\ B_1' &= B_1 , \qquad & B_2' &= \gamma\left(B_2 + \frac{v}{c^2}E_3\right) , \qquad & B_3' &= \gamma\left(B_3 - \frac{v}{c^2}E_2\right) . \end{aligned} \tag{7.29}$$

After a computation, in our current example with potentials (7.28) one finds

$$\mathbf{E}(\mathbf{r},t) \;=\; \frac{q}{4\pi\epsilon_0}\frac{\gamma}{\left[\gamma^2(x-vt)^2+x^2+z^2\right]^{3/2}}(\mathbf{r}-\mathbf{v}t) \;, \qquad \mathbf{B}(\mathbf{r},t) \;=\; \frac{1}{c^2}\,\mathbf{v}\times\mathbf{E} \;. \tag{7.30}$$

If we denote $\mathbf{R}\equiv\mathbf{r}-\mathbf{v}t$ to be the postion vector of the observation point $\mathbf{r}$, relative to the position vector $\mathbf{v}t$ of the point charge q, we may write these as

$$\mathbf{E} \;=\; \frac{q\gamma}{4\pi\epsilon_0}\,\frac{1}{\left[\gamma^2R_1^2+R_2^2+R_3^2\right]^{3/2}}\,\mathbf{R} \;, \qquad \mathbf{B} \;=\; \frac{\mu_0}{4\pi}\frac{q\gamma\,\mathbf{v}\times\mathbf{R}}{\left[\gamma^2R_1^2+R_2^2+R_3^2\right]^{3/2}} \;. \tag{7.31}$$

When $v \ll\; c$ we may approximate $\gamma\simeq 1$, and the equation for $\mathbf{E}$ is Coulomb's law (L1.7), while the equation for $\mathbf{B}$ is the Biot-Savart law (L3.10)! In particular, notice that we have effectively *derived* the Biot-Savart law from Coulomb's law, using only a Lorentz transformation!

* You may wonder what kind of objects transform as $\mathbf{E}$ and $\mathbf{B}$ in (7.29). They are clearly neither scalars nor four-vectors as $\vec{A}$. In special relativity they form *one object together*, the skew-symmetric matrix:

$$F_{ab} \;=\; \vec{\partial}_a\vec{A}_b-\vec{\partial}_b\vec{A}_a = \begin{pmatrix} 0 & \frac{E_1}{c} & \frac{E_2}{c} & \frac{E_3}{c} \\ -\frac{E_1}{c} & 0 & -B_3 & B_2 \\ -\frac{E_2}{c} & B_3 & 0 & -B_1 \\ -\frac{E_3}{c} & -B_2 & B_1 & 0 \end{pmatrix}_{ab} \;. \tag{7.32}$$

The existence of F makes it natural that $\mathbf{E}$ and $\mathbf{B}$ transmute into each other under Lorentz transformation.

There is of course much more to say about Special Relativity than the comments we have made in this section, but having derived magnetostatics from electrostatics and the structure of spacetime, we conclude here. ■