B1.2 Set Theory

Sheet 1 — HT26

For this sheet, assume only ZF1-6 (Extensionality, Empty Set, Pairing, Union, Powerset, Comprehension).

Section A

1. Show that for any sets x and y, there is a *unique* set whose elements are precisely x and y.

2. Let a be a set. Show that there is a set whose elements are precisely the two-element subsets of a.

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Section B

3. A set a is called **transitive** if $\bigcup a \subseteq a$. Prove:
   (a) A set a is transitive if and only if for all x and y, if $x \in y$ and $y \in a$, then $x \in a$.
   (b) $\emptyset$ is transitive.
   (c) If a is transitive then so is $a \cup \{a\}$.
   (d) a is transitive if and only if $\bigcup (a \cup \{a\}) = a$.
   (e) The intersection of any non-empty set of transitive sets is transitive.
   (f) The union of any set of transitive sets is transitive.
   (g) For any a, there is a set whose elements are precisely the transitive elements of a.

4. For each of the following statements, either prove that it holds for all sets a, b, or give a counterexample.
   (a) $\mathcal{P}\left(\bigcup a\right) = a$.
   (b) $\bigcup \mathcal{P}(a) = a$.
   (c) $\mathcal{P}(a) \subseteq \mathcal{P}(b)$ if and only if $a \subseteq b$.

5. Let a and b be sets. For each of the following, prove either that a set as described exists or that no such set exists.
   (a) $\{\{x, y\} : x \in a,\ y \in b\}$.
   (b) $\{x \cup y : x \in a,\ y \in b\}$.
   (c) $\{x : x \notin a\}$.
   (d) $\{x : a \in x\}$.
   (e) $\{\bigcup x : x \in a\}$.
   (f) $\{\{x, y\} : x \neq y\}$.

6. Suppose that there exist sets x and y such that $x = \{y\}$ and $y = \{x\}$. Deduce that there exists a non-empty set a with $a = \bigcup a$.

7. Consider the following potential universe of set theory. The objects are the natural numbers, and, writing $\in_a$ for the element relation in this universe, $n \in_a m$ if and only if the nth digit from the right of the binary expansion of m is 1. For example, the binary expansion of 13 is 1101, so $n \in_a 13$ precisely for $n = 0, 2, 3$.

   Show that ZF1-6 hold in this universe, assuming standard facts about the natural numbers.

   [This is known as the *Ackermann interpretation* of finite set theory in arithmetic.]

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