Solitary waves in thin-film models for fluid convection

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Mathematical Modelling and Scientific Computing (MMSC) Case Study Project

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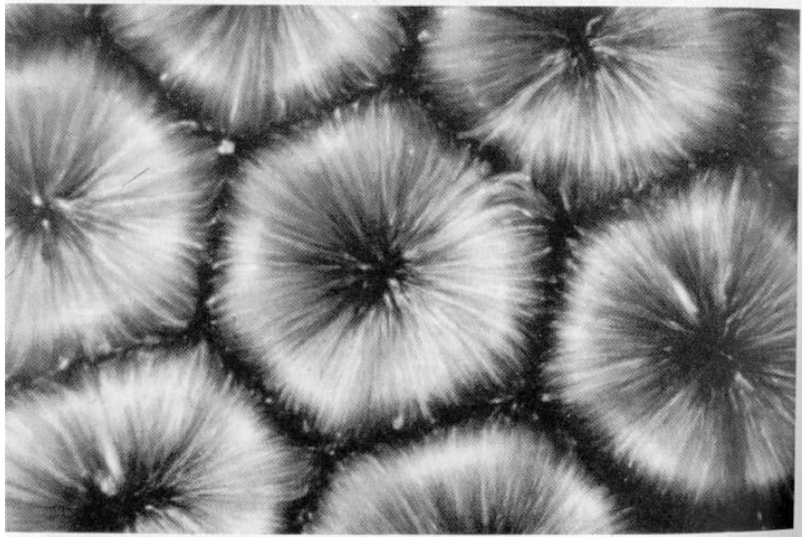
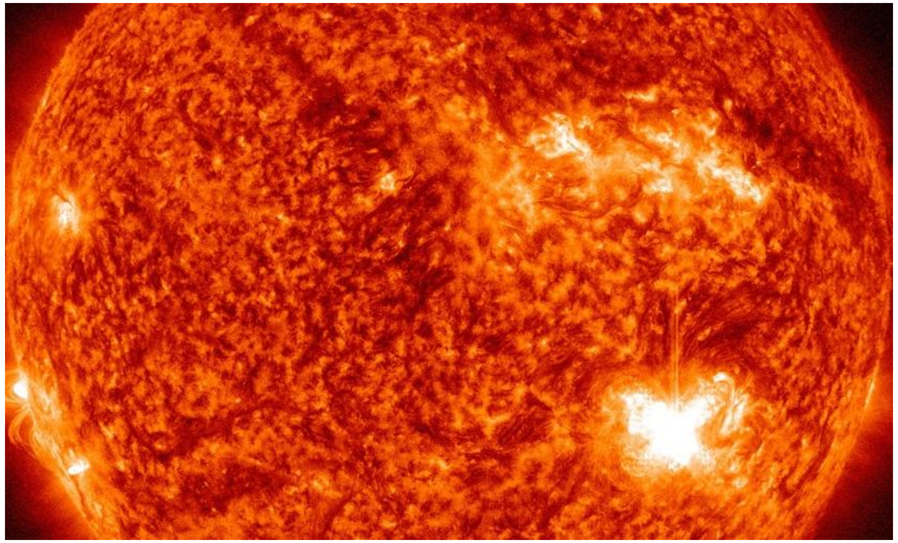
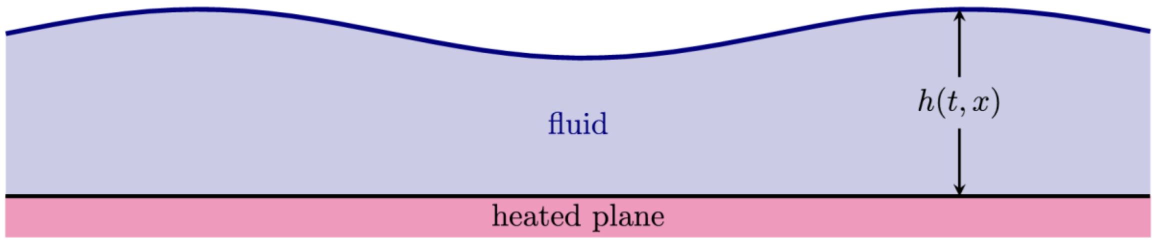
Fluid convection patterns

- Heating fluids from below can generate convective currents
- Local currents push back against their neighbours, forming regularly-spaced structures
- For lower viscosities, fluids exhibit turbulent flows, like what we see on the surface of the sun





Thin-film models

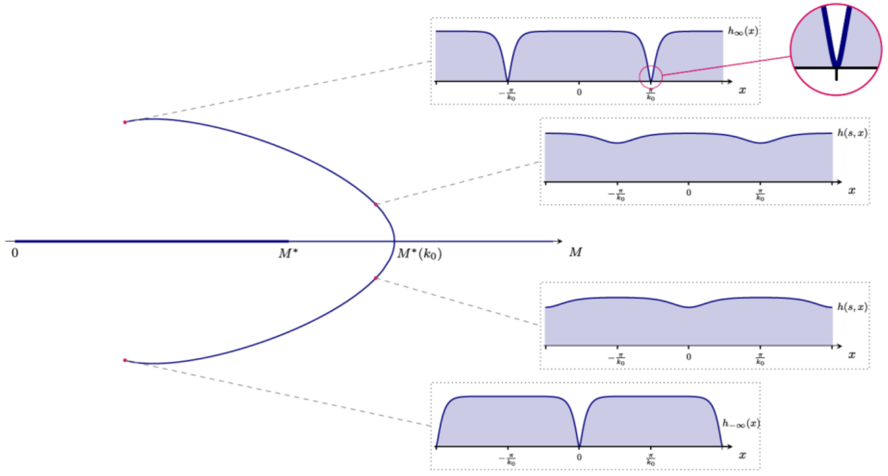


- Bénard-Marangoni convection – driven by changes in surface tension caused by temperature changes
- (1+3)-dimensional PDE problem with free surface
- Thin-film models derived via lubrication approximation for |h|<<1 and averaging temperature over the height of the fluid
- Model 1: $$\partial_t h + \nabla \cdot \left(h^3 \nabla(\Delta h - gh) + M \left(\frac{h}{1+h} \right)^2 \nabla h \right) = 0$$
- Model 2: $$\begin{aligned} \partial_t h + \nabla \cdot \mathbf{j}_1(h,\theta) = 0, \\ h\, \partial_t \theta + \nabla \cdot \mathbf{j}_2(h,\theta) = 0, \end{aligned}$$

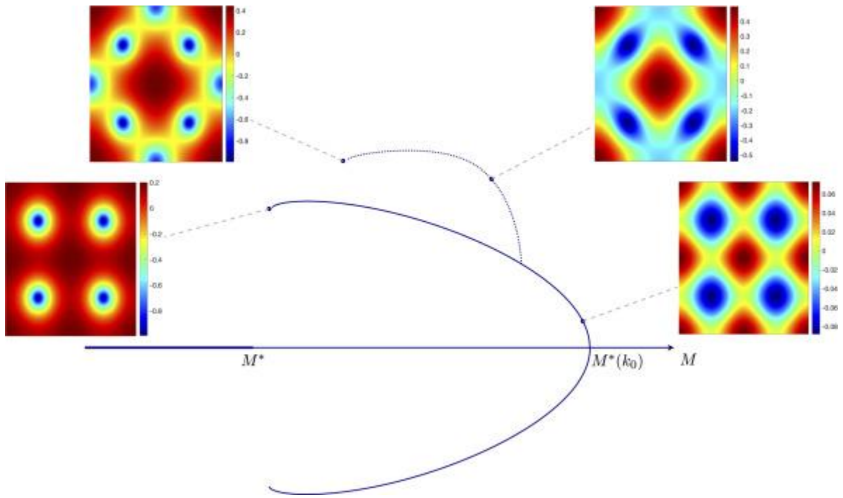
$h = h(t,x,y), \theta = \theta(t,x,y)$

h: height of fluid
θ: temperature
M: Marangoni number
g: gravitational constant

Thin-film Bénard-Marangoni convection - periodic

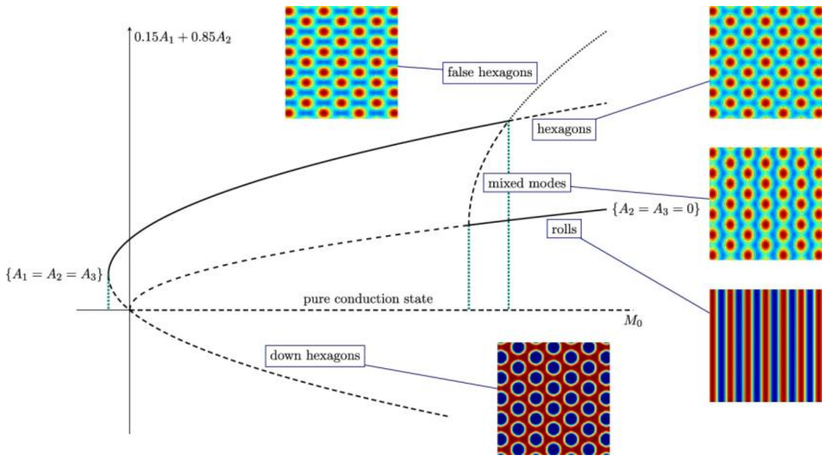


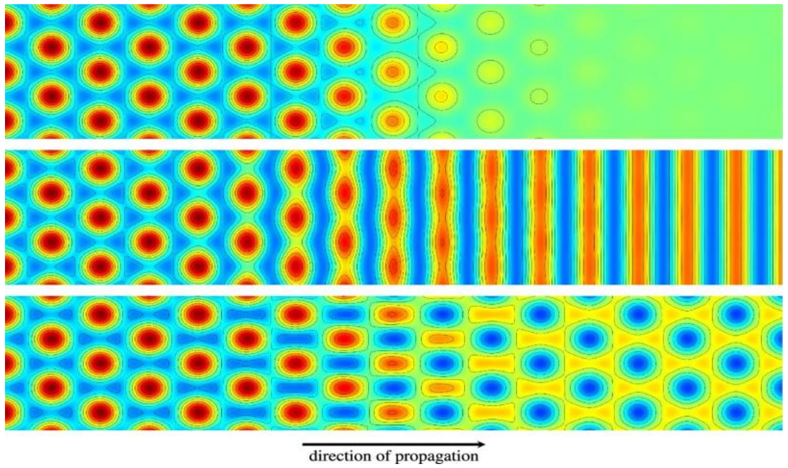
Bifurcations of periodic patterns in Model 1



G. Bruell, B. Hilder & J. Jansen, Nonlinearity (2024)

S. Böhmer, B. Hilder & J. Jansen, Physica D (2025)



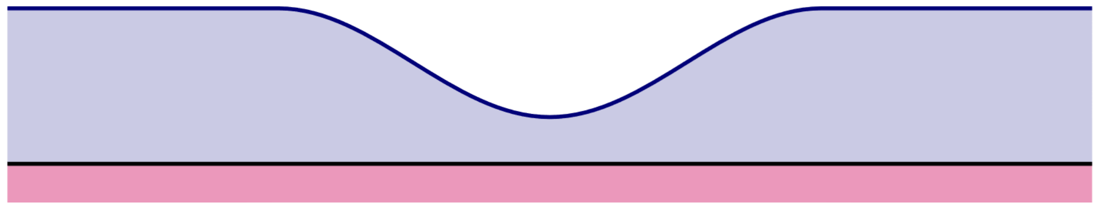


Bifurcations of periodic patterns and invading fronts in Model 2

B. Hilder & J. Jansen, JNLS (2025)

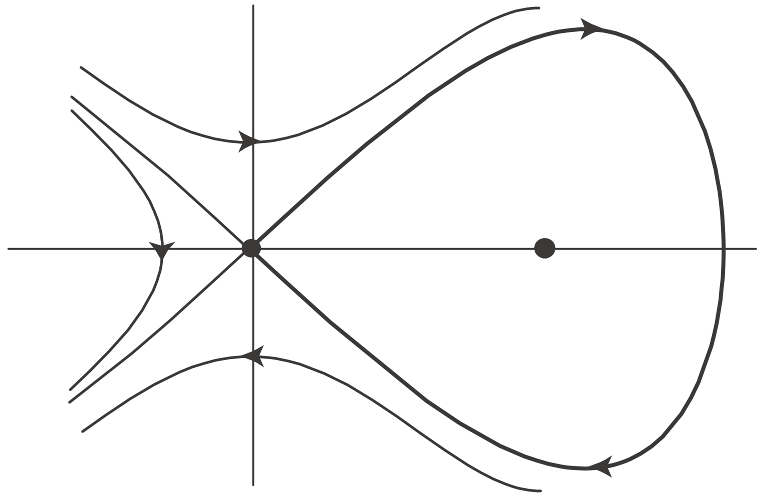
Solitary waves in Model 1

- Look for steady one-dimensional solitary waves in Model 1



$$h = h(x), \qquad h(x) \to \bar{h} \quad \text{constant as } x \to \pm\infty$$

- Model 1 reduces to a fourth-order ODE



- We want to reduce this ODE to a two-dimensional first-order ODE system and look for homoclinic orbits

Project Summary

Using ODE/dynamical systems theory to analyse a given model and prove the existence of solitary waves

Extensions: film rupture, stability, radial solitary waves, numerical simulations, etc.

Pre-requisites: ODE theory, perturbation methods