Non-equilibrium Flow of Water in Unsaturated Porous Media

Porous Medium Equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left[D(\theta) \frac{\partial \theta}{\partial x} \right]$$

D = diffusivity

Many applications

- Flow of water through soils
- Heat transfer
- Oil reservoirs
- Math biology
- Chemical engineering
- Pollutant transport in groundwater

Unsaturated Fluid flow through Soils.

This is governed by the well know Richards' equation

$$\frac{\partial \theta}{\partial t} = \underbrace{\frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right]}_{\substack{capillary \\ suction}} - \underbrace{\frac{\partial K(\theta)}{\partial z}}_{gravity}, \quad 0 \le \theta \le 1$$

u = soil saturation

The soil pressure P is usually measured at steady state conditions, and this is usually called the soil moisture characteristic curve $\psi(\theta) < 0$. Thus typically

$$P(\theta) = \psi(\theta)$$

FINGER PERSISTENCE IN UNSATURATED SOILS

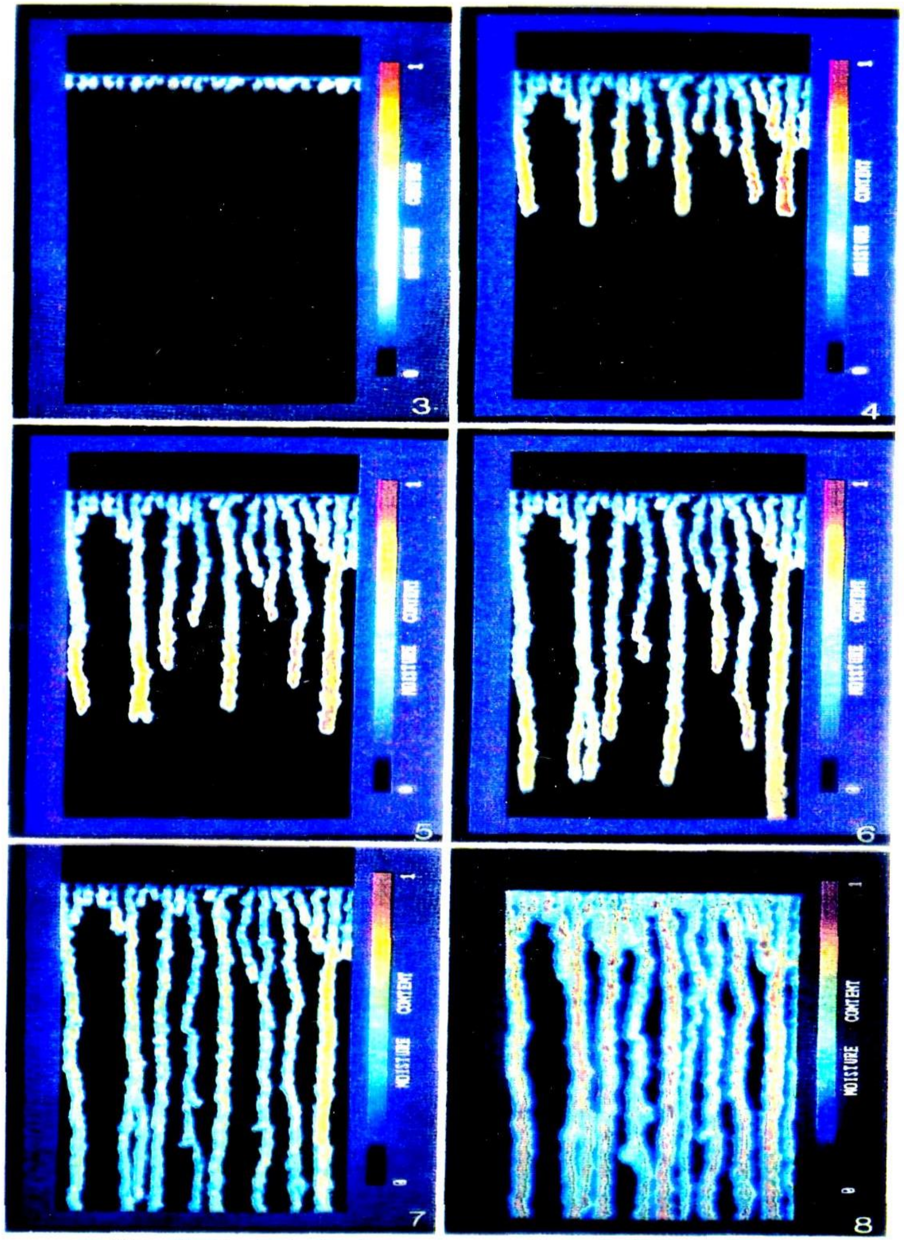
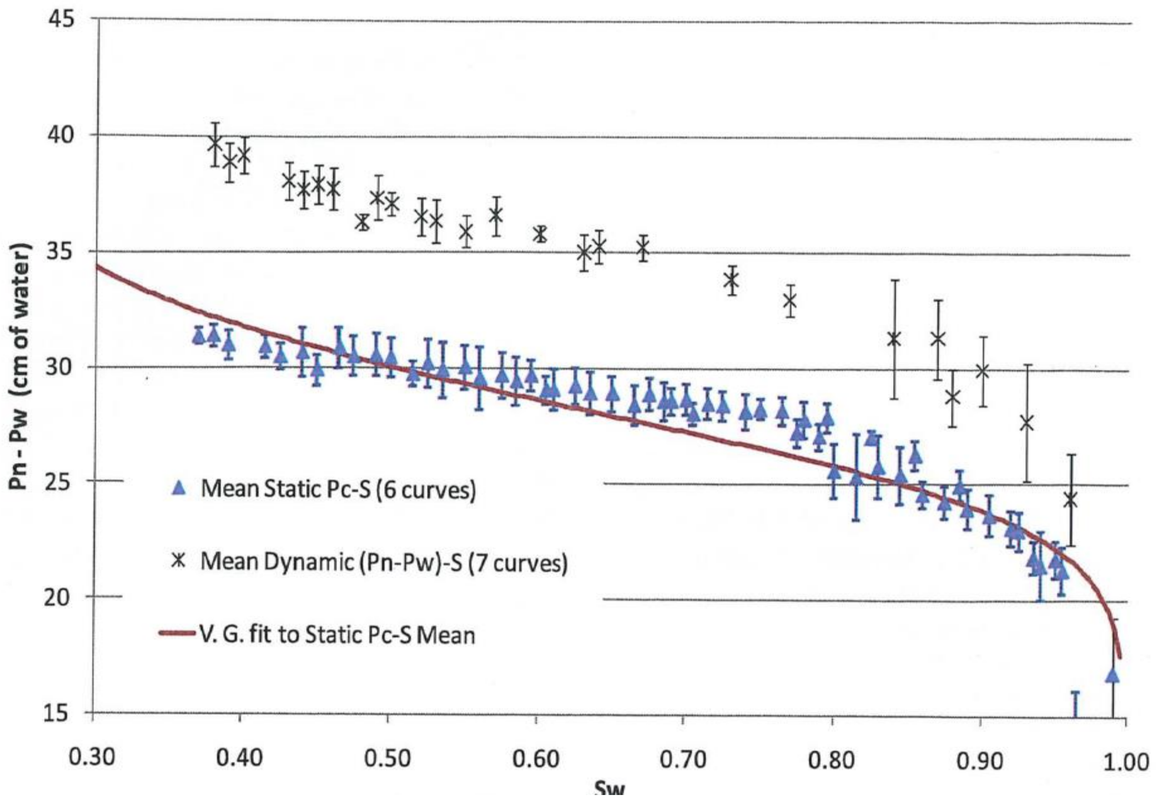


FIG. 3. Cycle 1, 2 min.
FIG. 4. Cycle 1, 4 min.
FIG. 5. Cycle 1, 5 min.
FIG. 6. Cycle 1, 6 min.
FIG. 7. Cycle 1, 15 min.
FIG. 8. Cycle 1, 2 d.

It has been proven that Richards' equation can only produce monotonic solutions from a monotonic boundary condition into an initially dry soil.

However, these finger-like profiles have non-monotonic saturations.

When *P* is measured under transient flow conditions



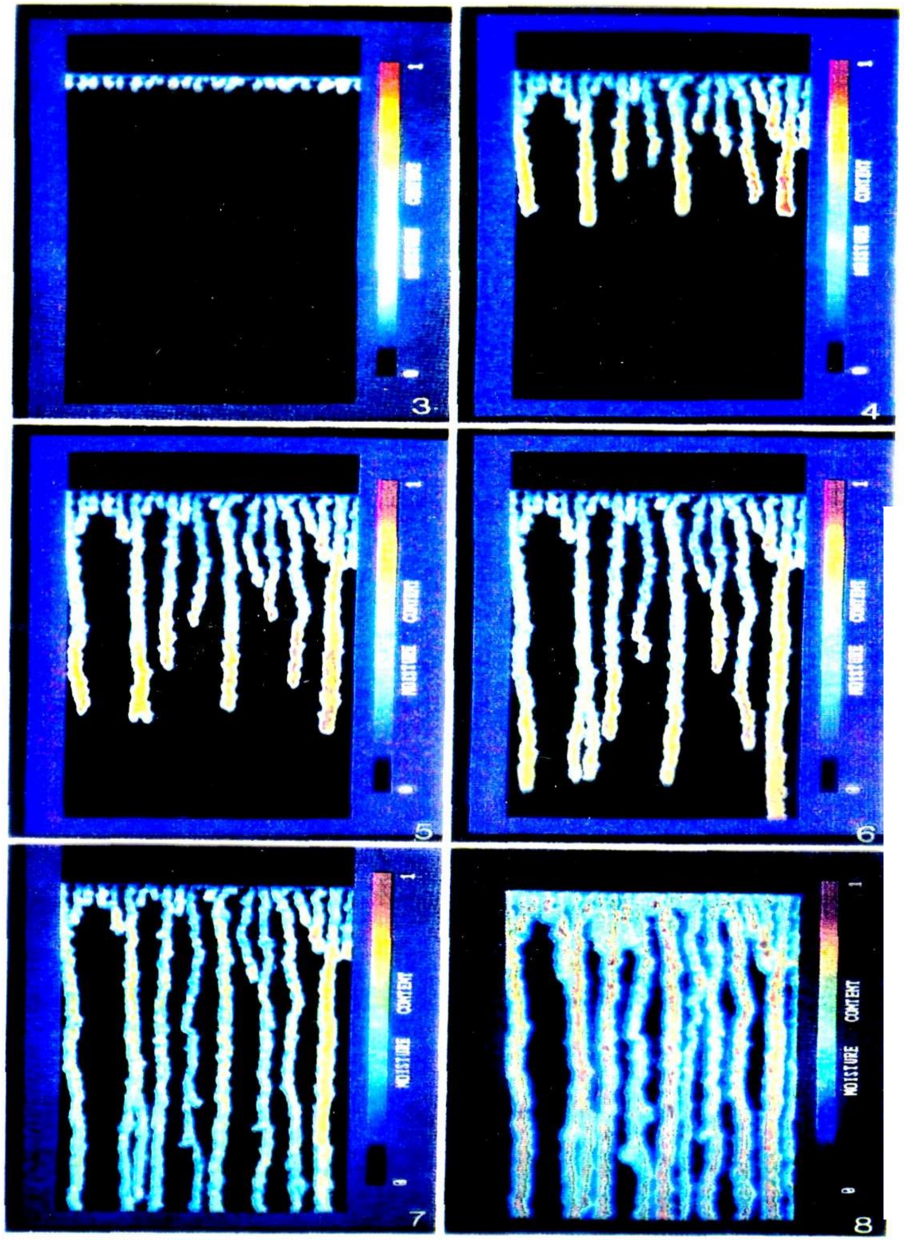
Nonequilibrium Richards Equation

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z} \right] + \underbrace{\frac{\partial}{\partial z}\left[K(\theta)\frac{\partial}{\partial x}\left(\tau(\theta)\frac{\partial \theta}{\partial t} \right) \right]}_{\substack{\text{additional} \\ \text{non-equilibrium} \\ \text{term}}} - \frac{\partial K(\theta)}{\partial z}$$

$$P(\theta) = \psi(\theta) + \underbrace{\tau(\theta)}_{\substack{\text{relaxation} \\ \text{function}}} \frac{\partial \theta}{\partial t}$$

$\tau(\theta)$ is a measure of the time for the pore pressure to equilibrate to a change in saturation.

FINGER PERSISTENCE IN UNSATURATED SOILS



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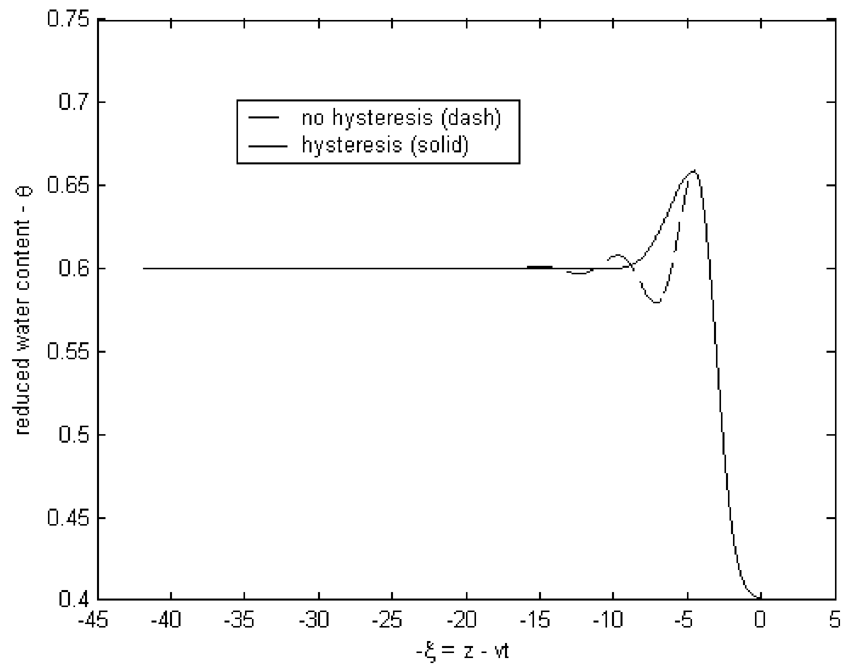
Travelling wave solutions are able to reproduce the non-monotonic behaviour

$$\theta(z,t) = f(z+vt)$$



Similarity Solutions

Can we find other types of similarity solutions that are not travelling wave?

To simplify slightly – consider only horizontal flow

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left[D(\theta)\frac{\partial \theta}{\partial z}\right] + \frac{\partial}{\partial z}\left[K(\theta)\frac{\partial}{\partial x}\left(\tau(\theta)\frac{\partial \theta}{\partial t}\right)\right]$$

$$D = \theta^{\alpha}$$

$$K = \theta^{\beta}$$

$$\tau = \theta^{r}$$

Traditional similarity solution forms to look for are $\theta = t^{\gamma} f(\eta), \quad \eta = \frac{x}{t^{\omega}}$

Do such solutions exist that are physically sensible, and if so, how do they depend on the parameters? Do they apply for sensible boundary and initial conditions?