Lie groups. C3.5. HT26 [P. Bousseau]

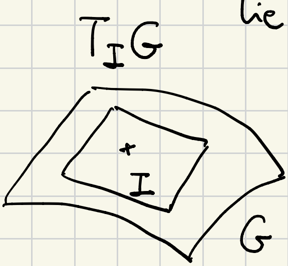
Lecture 1. [19/01/2026]

1. Introduction.

- Groups: "symmetries" of geometric objects.
- Some groups have their own geometric structure.
  "Lie groups", named after Sophus Lie (1842-1899)
- ↳ Groups whose elements depend on finitely many continuous parameters.
- At the intersection of:
  - Algebra
  - Geometry
  - Analysis!

Highlights / Plan of the course

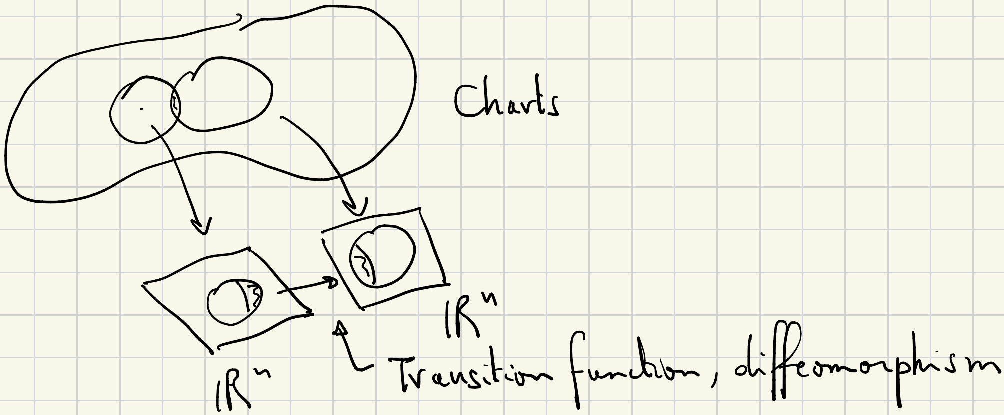
- Interplay between Lie groups and Lie algebras. Linearization, even more powerful than in ordinary calculus due to the group structure



- Representation theory
- Compact Lie groups, maximal tori, Weyl groups (finite groups)
- State (no proof) Classification of connected compact Lie groups. Exceptional Lie groups (eg G_2)

2. Fundamentals.

Recall: • n-dim (smooth) // C^∞ manifolds: topological space + local identifications with $\mathbb{R}^n$



Ex: $\mathbb{R}^n$, open in $\mathbb{R}^n$, S^n, $T^n = (S^1)^n$

M, N manifolds $\Rightarrow$ $M \times N$ manifold $\dim = \dim M + \dim N$

• Smooth map $f: M \to N$ between manifolds (smooth in charts)

Def: A Lie group G is a manifold endowed with a structure of group, s.t. the multiplication $m: G \times G \to G$ and the inversion $i: G \to G$ are smooth.

Ex: • $(\mathbb{R}^n, +)$ abelian Lie group

• $GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \mid \det(A) \neq 0\}$ $\times$ of matrices

Group of invertible $n \times n$ matrices.

Open in $M(n,\mathbb{R}) \simeq \mathbb{R}^{n^2}$ → Manifold of dim n^2

Product: polynomial so smooth

Inverse? $A^{-1} = \dfrac{1}{\underbrace{\det(A)}_{\text{polynomial}}} \underbrace{\operatorname{adj}(A)}_{\text{adjugate matrix}}$

[Transpose of the matrix of cofactors

$(-1)^{i+1}$ $\underbrace{\text{minor}}$

$(n-1)\times(n-1)$ determinant

→ Polynomial

$\dfrac{\text{Polynomial}}{\text{Polynomial}}$

so smooth → Lie group

Tangent space at the identity: $T_I\, GL(n,\mathbb{R}) = M(n,\mathbb{R})$

More examples? Recall: <u>Regular Value Theorem</u>:

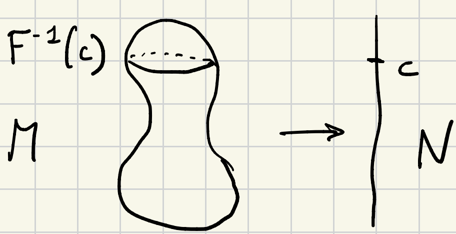
Let M, N be manifolds of dim m, n and $F: M \to N$ smooth.

Let $c \in N$ be a regular value of M with $F^{-1}(c) \neq \emptyset$

[$\forall p \in F^{-1}(c)$, $dF_p : T_pM \longrightarrow T_{f(p)=c}N$ is surjective]

Then $F^{-1}(c)$ is a manifold of dim $m-n$,

and $\forall p \in F^{-1}(c)$, $T_p F^{-1}(c) = \operatorname{Ker}(dF_p)$



Useful to construct manifolds and identify tangent spaces.

Examples: $SL(n,\mathbb{R}) = \{A \in M(n,\mathbb{R}) \mid \det(A) = 1\}$

$F : M(n,\mathbb{R}) \to \mathbb{R}$ $\qquad SL(n,\mathbb{R}) = F^{-1}(0)$

$A \mapsto \det|A| - 1$ $\qquad A \in SL(n,\mathbb{R})$

$$d(\det)_A(B) = tr(A^{-1}B)$$

$$M(n,\mathbb{R}) \to \mathbb{R}$$
$$B \mapsto tr(A^{-1}B)$$

surjective

eg $B = \frac{\lambda A}{n}$

$tr(A^{-1}B) = \lambda$

so $SL(n,\mathbb{R})$ manifold $\to$ lie group

and $T_I SL(n,\mathbb{R}) \simeq \{B \mid tr(B) = 0\}$

$$\dim = n^2 - 1$$

- $\left.\begin{array}{l} S^1 = \{z \mid |z| \in \mathbb{C}\} \\ T^n = (S^1)^n \end{array}\right\}$ compact abelian lie groups
- $O(n) = \{A \in M(n,\mathbb{R}) \mid A^T A = I\}$

$F : M(n,\mathbb{R}) \longrightarrow Sym(n,\mathbb{R})$ $\qquad O(n) = F^{-1}(I)$

$A \longmapsto A^T A$

$dF_A(B) = B^T A + A^T B$ $\qquad$ surjective $[B = \frac{1}{2}AC]$

$\to O(n)$ lie group

$T_I O(n) = \{B \in M(n,\mathbb{R}) \mid B^T = -B\}$

$\dim = \frac{n(n-1)}{2}$ $\qquad$ skew-symmetric matrix

- $SO(n) = \{A \in O(n) \mid \det(A) = 1\}$

  Connected component of $O(n)$

- $U(n) = \{A \in M(n, \mathbb{C}) \mid \underbrace{A^*}_{= \overline{A}^T} A = I\}$ dim $= n^2$

$T_I U(n) = \{B \in M(n, \mathbb{C}) \mid B^* = -B\}$

anti-Hermitian

$SU(n) = \{A \in U(n) \mid \det(A) = 1\}$ dim $n^2 - 1$

$T_I SU(n) = \{B \in M(n, \mathbb{C}) \mid B^* = -B, \ \mathrm{tr}(B) = 0\}$

<u>Def</u>: G, H Lie groups. • $f: G \to H$ Lie group homomorphism if:

• f smooth, • f group hom.

• $f: G \to H$ Lie group isom if:

• f diffeo, • f group isom.

<u>Ex</u>: $\det: GL(n, \mathbb{R}) \to \mathbb{R} \setminus \{0\}$ Lie group hom.

$\det: U(n) \to S^1$ Lie group hom.