C4.1 Further Functional Analysis

Sheet 4 — MT 2025

For classes in week 1 HT

Sections A and B are based on material up to and including Section 10. Section C contains some questions based on material from Sections 12 and 13.

Section A

1. Let X and Y be normed spaces $T \in \mathcal{B}(X,Y)$. Fill in the details required to show that T is compact if and only if for every bounded sequence $(x_n)_{n=1}^{\infty}$, there is a subsequence $(x_{n_k})_k$ such that $(Tx_{n_k})_k$ converges.

   Solution: If T is compact, and x_n is a sequence bounded by M, then $T(x_n/M)$ is a sequence in the compact metric space $\overline{T(B_X)}$ so has a convergent subsequence, say $T(x_{n_k}/M)$. Hence $T(x_{n_k})_k$ converges.

   Conversely, if the condition holds, then given a sequence $(y_n) \in \overline{T(B_X)}$, choose $x_n \in B_X$ with $\|Tx_n - y_n\| < 1/n$. Then x_n has a subsequence (x_{n_k}) so that $T(x_{n_k})$ converges, and hence too (y_{n_k}) converges. Thus $\overline{T(B_X)}$ is sequentially compact, so compact (as it is a metric space).

2. Show that c_0 embeds isometrically into $\mathcal{K}(\ell^2)$. Deduce that $\mathcal{K}(\ell^2)$ is not reflexive.

   Solution: For $(a_n)_n \in c_0$ define $T \in \mathcal{B}(\ell^2)$ by $T(x_n) = (a_n x_n)$. It is not difficult to confirm that this is an isometry (the bound from above is immediate; then find an element where the supremum is achieved). One may then define a finite-rank approximation of T via $T^N(x_n) = (a_n x_n)$ for all $n \leq N$, and $T^N(x_n) = 0$ for all $n > N$. Choosing $x \in B_{\ell^2}$ we have

$$\|T - T^N\| \leq \max_{n \geq N+1} |a_n| \to 0 \text{ as } N \to \infty.$$

   By Theorem 9.4, T maps into the compact operators. Since reflexivity passes to closed subspaces (Theorem 6.8; see also Sheet 2 problem B.7), and c_0 is not reflexive, neither is $\mathcal{K}(\ell^2)$.

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3. This question aims to revise your knowledge of the spectrum of self-adjoint operators on a Hilbert space. If you've not seen this before, then the later parts won't be warm up exercises. Let X be a Hilbert space and $T \in \mathcal{B}(X)$.

   (a) Show that if T is self-adjoint, all eigenvectors are real and eigenvectors corresponding to distinct eigenvalues are orthogonal.

   (b) Show that T is surjective if and only if the adjoint $T^\star$ is bounded below. Use this to show that if $\lambda \in \sigma(T)$ then there is a sequence $(x_n)_{n=1}^\infty$ in S_X such that $(x_n, Tx_n) \to \lambda$.

   (c) If T is self-adjoint show that $\|T\| = \sup_{x \in S_X} |(x, Tx)|$ and deduce that $r(T) = \|T\|$.

Solution:

(a) This is as it was in prelims / part A! Let x be an eigenvector with eigenvalue λ. Then

$$\overline{\lambda}\|x\|^2 = \langle Tx, x\rangle = \langle x, Tx\rangle = \lambda\|x\|^2.$$

Since $x \neq 0$, $\overline{\lambda} = \lambda$, and $\lambda \in \mathbb{R}$.

Suppose $y \in X$ is an eigenvector with eigenvalues μ. As

$$\lambda\langle x, y\rangle = \langle Tx, y\rangle = \langle x, Ty\rangle = \mu\langle x, y\rangle,$$

if $\lambda \neq \mu$, then $\langle x, y\rangle = 0$.

(b) Suppose T is surjective, then T is a quotient operator so T^* is an isomorphic embedding so bounded below. Hence $T^\star$ is also bounded below (from the connection between the dual operator $T^* : H^* \to H^*$ and the Hilbert space adjoint $T^\star : H \to H$.) Conversely if $T^\star$ is bounded below, then it is an isomorphic embedding and hence so too is T^*. Then T is a quotient operator, so surjective.

Suppose $\lambda \in \sigma(T)$. Either $T - \lambda I$ is not injective, when there exists $x \in S_X$ with $Tx = \lambda x$, or $T - \lambda I$ is not surjective. Thus $(T - \lambda I)^\star = T^\star - \overline{\lambda} I$ is not bounded below, and hence there exists a sequence $(x_n)_n$ in S_X with $\|(T - \lambda I)^\star x_n\| \to 0$. Thus $\langle T^\star x_n - \overline{\lambda} x_n, x_n\rangle \to 0$ and so $\langle x_n, Tx_n\rangle \to \lambda$.

(c) Recall that for $T \in \mathcal{B}(X)$, we have $\|T\| = \sup_{x,y \in S_X} |\langle Tx, y\rangle|$.

Now, for $T = T^\star$, note that for $x, y \in S_X$,

$$\begin{aligned}\langle T(x+y), x+y\rangle - \langle T(x-y), (x-y)\rangle = 2\langle Tx, y\rangle + 2\langle Ty, x\rangle &= 2\langle Tx, y\rangle + 2\langle x, Ty\rangle \\ &= 4\Re\langle Tx, y\rangle.\end{aligned}$$

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Now, writing $M = \sup_{x \in S_X} |\langle Tx, x\rangle|$, we have

$$\begin{aligned}\Re\langle Tx, y\rangle &= \frac{1}{4}(\langle T(x+y), x+y\rangle - \langle T(x-y), (x-y)\rangle) \\ &\leq \frac{M}{4}(\|x+y\|^2 + \|x-y\|^2) = \frac{M}{4}(2\|x\|^2 + \|y\|^2) \leq M,\end{aligned}$$

using the parallelogram law. Multiplying by a suitable scalar of norm 1 we get $|\langle Tx, y\rangle| \leq M$.

We have $r(T) \leq \|T\|$ for all bounded operators T. Now let $T = T^\star$ and choose a sequence $x_n \in S_X$ with $|\langle Tx_n, x_n\rangle| \to \|T\|$. We can pass to a subsequence such that $\langle Tx_n, x_n\rangle \to \lambda = \pm\|T\|$. Now (as $\langle Tx_n, x_n\rangle, \lambda$ are both real),

$$\begin{aligned}0 \leq \|(T-\lambda)x_n\|^2 &= \langle (T-\lambda)x_n, (T-\lambda)x_n\rangle \\ &= \|Tx_n\|^2 - 2\lambda\langle Tx_n, x_n\rangle + \lambda^2 \leq 2\lambda^2 - 2\lambda\langle Tx_n, x_n\rangle \to 0.\end{aligned}$$

Hence $T - \lambda I$ is not bounded below, and hence $\lambda = \pm\|T\| \in \sigma(T)$. Thus $r(T) \leq \sigma(T)$.

4. Let X be a separable Hilbert space. An operator $T \in \mathcal{B}(X)$ is a *Hilbert-Schmidt* operator if there is an orthonormal basis $(e_n)_{n=1}^\infty$ for X such that $\sum_{n=1}^\infty \|T(e_n)\|^2 < \infty$.

   (a) Show that if $(e_n)_{n=1}^\infty$ and $(f_m)_{m=1}^\infty$ are orthonormal bases for X, then $\sum_m \|T(f_m)\|^2 = \sum_n \|T(e_n)\|^2$ for any $T \in \mathcal{B}(X)$.

   Solution: For each $m \in \mathbb{N}$, write $f_m = \sum_n \alpha_n^m e_n$, where $\alpha_n^m \in \mathbb{F}$ is zero for all but finitely many values of n. Then

$$\begin{aligned}\sum_m \|Tf_m\|^2 &= \sum_m \left\| T\left(\sum_n \alpha_n^m e_n\right)\right\|^2 \\ &= \sum_m \left\langle \sum_i \alpha_i^m Te_i, \sum_j \alpha_j^m Te_j \right\rangle \\ &= \sum_m \sum_{i,j} \alpha_i^m \overline{\alpha_j^m} \langle Te_i, Te_j\rangle \\ &= \sum_{i,j} \langle Te_i, Te_j\rangle \sum_m \alpha_i^m \overline{\alpha_j^m} \\ &= \sum_{i,j} \langle Te_i, Te_j\rangle \sum_m \alpha_i^m \overline{\alpha_j^m}.\end{aligned}$$

   We now prove that $\sum_m \alpha_i^m \overline{\alpha_j^m} = \delta_{ij}$, which will conclude the proof of this part. Write $e_n = \sum_m \beta_m^n f_m$. Then $\langle e_n, f_m\rangle = \beta_m^n = \alpha_n^m$, so

$$\sum_m \alpha_i^m \overline{\alpha_j^m} = \sum_m \beta_m^i \overline{\beta_m^j} = \langle e_i, e_j\rangle = \delta_{ij},$$

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as desired.

(b) Show that every Hilbert-Schmidt operator is compact.

Solution: Consider the sequence of operators $T_n \in \mathcal{B}(X)$ defined by $T_n(e_i) = Te_i$ for $i \leqslant n$ and $T_n(e_i) = 0$ for $i > n$ and extend to all of X by density. Note that $T_n(X)$ is the closure of $T_n|_{\mathrm{span}\{e_i\}}(X)$. But $T_n|_{\mathrm{span}\{e_i\}}(X)$ is finite dimensional and therefore closed. Thus, the operators T_n have finite rank. We now show that $\lim_{n\to\infty} \|T - T_n\| = 0$. Let ε and let $n \in \mathbb{N}$ be such that $\sum_{i>n} \|Te_i\|^2 < \varepsilon$, which is possible since T is Hilbert–Schmidt. Let $x = \sum_i \alpha_i e_i \in B_X$. Then

$$\|(T - T_n)(x)\|^2 = \left\| \sum_{i>n} \alpha_i T e_i \right\|^2 \leqslant \sum_{i>n} |\alpha_i|^2 \cdot \|Te_i\|^2 \leqslant \sum_{i>n} \|Te_i\|^2 < \varepsilon.$$

This proves the claim and shows that T is compact by Corollary 11.4 of the notes.

(c) Give a characterisation in terms of eigenvalues and multiplicities of when a compact self-adjoint operator is Hilbert-Schmidt.

Solution: Let T be a compact self-adjoint operator. By the Spectral Theorem, there are nonzero real numbers λ_i and finite-rank orthogonal projections P_i such that $T = \sum_i \lambda_i P_i$ (the sum is either finite or countably infinite) and there is a basis E of orthonormal eigenvectors. For each i, let $e_1^i, \ldots, e_{m(i)}^i$ be the eigenvectors of E with eigenvalue to λ_i. Note that $m(i)$ is the multiplicity of λ_i, and it is always finite (by Theorem 13.3, for example).

$$\begin{aligned}
\sum_{e\in E} \|Te\|^2 &= \sum_i \sum_{k=1}^{m(i)} \left\|Te_k^i\right\|^2 \\
&= \sum_i \sum_{k=1}^{m(i)} \left\langle \sum_j \lambda_j P_j e_k^i, \sum_j \lambda_j P_j e_k^i \right\rangle \\
&= \sum_i \sum_{k=1}^{m(i)} |\lambda_i|^2 \left\langle \sum_{k=1}^{m(i)} e_k^i, \sum_{k=1}^{m(i)} e_k^i \right\rangle \\
&= \sum_i \sum_{k=1}^{m(i)} |\lambda_i|^2 m(i) \\
&= \sum_i |\lambda_i|^2 m(i)^2.
\end{aligned}$$

We thus obtain the following characterisation: T is Hilbert–Schmidt if and only if

$$\sum_i |\lambda_i|^2 m(i)^2 < \infty.$$

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Section B

1. (a) Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X, Y)$. We say that T is *completely continuous* if, for every weakly convergent sequence (x_n) in X, the sequence (Tx_n) is norm-convergent in Y.

   (i) Show that if T is compact then T is completely continuous.

   (ii) Prove that the converse of (i) holds if X is reflexive. [*You may, if you wish, assume in addition that X is separable.*]

   (iii) Exhibit an operator which is completely continuous but not compact.

   (b) Let $1 < p < \infty$. Show that $\mathcal{B}(\ell^p, \ell^1) = \mathcal{K}(\ell^p, \ell^1)$. Is $\mathcal{B}(c_0, \ell^p) = \mathcal{K}(c_0, \ell^p)$?

Solution:

(a) (i) Note first that for $T \in \mathcal{B}(X, Y)$ we have $T^*(Y^*) \subseteq X^*$, so any operator $T \in \mathcal{B}(X, Y)$ is weakly continuous. If T is compact then the norm closure M of $T(B_X)$ is compact. Since the weak topology is Hausdorff and coarser than the norm topology the identity map on M is norm-to-weak-continuous and hence a homeomorphism for these topologies, so the two topologies coincide on M. Suppose that (x_n) is a weakly convergent sequence in X. By the Uniform Boundedness Principle the sequence (x_n) is bounded, so we may assume that its elements lie in B_X. Now the sequence (Tx_n) is weakly convergent by weak continuity of T. Since the weak topology coincides on M with the norm topology we deduce that (Tx_n) is norm-convergent. Thus T is completely continuous.

Alternatively, one may prove the result as follows. Suppose T is compact but is not completely continuous. Then there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset B_X$ such that $x_n \rightharpoonup x \in X$ while the sequence (Tx_n) does not converge in Y. By the compactness of T, there exists a subsequence (Tx_{n_k}) such that $Tx_{n_k} \to y$ for some $y \in Y$. By the uniqueness of weak limits, we have $x_{n_k} \rightharpoonup x$ and so there must hold $Tx = y$. By the standard *subsequence of a subsequence* argument, we conclude that *every* subsequence of Tx_n converges to the same limit, and in particular, $Tx_n \to Tx = y$. This contradicts the assumption that (Tx_n) does not converge in Y; hence, T cannot be compact while not completely continuous, and so we conclude that compactness of T implies complete continuity of T.

(ii) Now suppose that X is reflexive and let (x_n) be a sequence in B_X. Let Z denote the closed linear span of $\{x_n : n \geq 1\}$. Then Z is separable and, being a closed subspace of a reflexive space, Z is reflexive. Hence the unit ball B_Z

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is weakly compact. Moreover, the bidual Z^{**} of Z is separable and hence so is Z^*. It follows that the weak topology on B_Z is compact and metrisable, so we may find a subsequence (x_{n_k}) of (x_n) which converges with respect to the weak topology on Z. But then (x_{n_k}) also converges with respect to the weak topology on X, so by complete continuity (Tx_{n_k}) converges in norm. Hence T is compact. Note that instead of considering the space Z we could have appealed to the Eberlein-Šmulian Theorem (had we included that result in the course - but we didn't).

(iii) Let $X = Y = \ell^1$ and let T be the identity operator on ℓ^1. Since ℓ^1 has the Schur property any weakly convergent sequence in ℓ^1 is norm-convergent, so T is completely continuous. However, T is not compact because ℓ^1 is infinite-dimensional.

(b) Suppose that $T \in \mathcal{B}(\ell^p, \ell^1)$ and that (x_n) is a weakly convergent sequence in ℓ^p. By weak continuity of T the sequence (Tx_n) is weakly convergent in ℓ^1, and since ℓ^1 has the Schur property it follows that (Tx_n) converges in norm. Hence T is completely continuous. Since ℓ^p is reflexive for $1 < p < \infty$, T is compact by part (a). Hence $\mathcal{B}(\ell^p, \ell^1) = \mathcal{K}(\ell^p, \ell^1)$.

The answer to the last part is yes. Indeed, suppose that $T \in \mathcal{B}(c_0, \ell^p)$. If we dentify the dual spaces of c_0 and ℓ^p with ℓ^1 and ℓ^q, respectively, where $q \in (1, \infty)$ is the Hölder conjugate of p, then $T^* \in \mathcal{B}(\ell^q, \ell^1)$. Thus T^* is compact by the previous part, and by Schauder's Theorem and completeness of ℓ^p the operator T is also compact. Hence $\mathcal{B}(c_0, \ell^p) = \mathcal{K}(c_0, \ell^p)$.

Remark: The results in part (b) are special cases of Pitt's Theorem, which states that $\mathcal{B}(X, \ell^q) = \mathcal{K}(X, \ell^q)$ whenever $1 \leq q < \infty$ and either $X = \ell^p$ for $q < p < \infty$ or $X = c_0$. Since isomorphisms between infinite-dimensional normed vector spaces are never compact operators we see, in particular, that none of the classical sequence spaces in question are isomorphic.

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2. Let $K \in L^2(\mathbb{R}^2)$ and consider the map T sending $x \in L^2(\mathbb{R})$ to the function Tx defined by

$$(Tx)(t) = \int_{\mathbb{R}} K(s,t)x(s)\,\mathrm{d}\,s$$

whenever $t \in \mathbb{R}$ is such that the integral exists.

(a) Show that T is a well-defined element of $\mathcal{B}(L^2(\mathbb{R}))$ with $\|T\| \le \|K\|_{L^2(\mathbb{R}^2)}$.

(b) Prove that T is compact. [*You may use the fact that indicator functions of bounded rectangles span a dense subspace of* $L^2(\mathbb{R}^2)$.]

Solution:

(a) By Fubini's Theorem the function $s \mapsto |K(s,t)|^2$ is integrable for almost all $t \in \mathbb{R}$ and the function $t \mapsto \int_{\mathbb{R}} |K(s,t)|^2\,\mathrm{d}s$ is integrable, with

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |K(s,t)|^2\,\mathrm{d}s\,\mathrm{d}t = \|K\|_2^2.$$

The function $s \mapsto K(s,t)x(s)$ is measurable for every $t \in \mathbb{R}$, and by Hölder's inequality we have

$$\left|\int_{\mathbb{R}} K(s,t)x(s)\,\mathrm{d}s\right|^2 \le \left(\int_{\mathbb{R}} |K(s,t)x(s)|\,\mathrm{d}s\right)^2 \le \|x\|_2^2 \int_{\mathbb{R}} |K(s,t)|^2\,\mathrm{d}s$$

for almost all $t \in \mathbb{R}$. Hence the integral defining Tx exists for almost all $t \in \mathbb{R}$. The previous estimate also shows that $Tx \in L^2(\mathbb{R})$ with $\|Tx\|_2 \le \|K\|_2\|x\|_2$. Since T is certainly linear it is indeed a well-defined bounded linear operator on $L^2(\mathbb{R})$ which satisfies $\|T\| \le \|K\|_2$.

(b) Suppose that $K = \mathbb{1}_{E\times F}$, where $E, F \subseteq \mathbb{R}$ are bounded intervals. Then

$$Tx = \left(\int_E x(t)\,\mathrm{d}t\right)\mathbb{1}_F, \quad x \in L^2(\mathbb{R}).$$

Recall that $L^{\text{step}}(\mathbb{R}^2) = \text{Span}\{\mathbb{1}_{E\times F} : E, F \subseteq \mathbb{R} \text{ are bounded intervals}\}$. Extending the above argument we see that for $K \in L^{\text{step}}(\mathbb{R}^2)$ the operator T corresponding to K has finite rank. Now suppose that $K \in L^2(\mathbb{R}^2)$ is given and let T be the corresponding operator. Given $\varepsilon > 0$ we may find $K_\varepsilon \in L^{\text{step}}(\mathbb{R}^2)$ such that $\|K - K_\varepsilon\|_2 < \varepsilon$. Let T_ε be the finite-rank operator corresponding to K_ε. It follows from the estimate in (a) that $\|T - T_\varepsilon\| \le \|K - K_\varepsilon\|_2 < \varepsilon$. Thus T lies in the norm closure of the set of finite-rank operators, so by completeness of $L^2(\mathbb{R})$ the operator T is compact.

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3. Let X and Y be normed vector spaces and let $T \in \mathcal{B}(X, Y)$. We say that T is *weakly compact* if the weak closure of $T(B_X)$ is weakly compact.
   (a) Show that T is weakly compact if and only if $\operatorname{Ran} T^{**} \subseteq J_Y(Y)$.
   (b) Prove that if T is weakly compact then T^* is weakly compact, and that if Y is complete then the converse holds too.

Solution:

(a) Suppose that T is weakly compact, that is, $M = \overline{T(B_X)}^w$ is weakly compact in Y. Since J_Y is weak-to-weak*-continuous, $J_Y(M)$ is weak*-compact in Y^{**} and because the weak* topology is Hausdorff, the set $J_Y(M)$ is in particular weak*-closed. From Goldstine's theorem we have

$$B_{X^{**}} = \overline{J_X(B_X)}^{w^*}$$

and since $T^{***}(J_{Y^*}(Y^*)) = J_{X^*}(T^*(Y^*)) \subseteq J_{X^*}(X^*)$ we infer from Proposition 7.7(2) that T^{**} is weak*-continuous, hence

$$T^{**}(B_{X^{**}}) = T^{**}\left(\overline{J_X(B_X)}^{w^*}\right) \subseteq \overline{T^{**}(J_X(B_X))}^{w^*}.$$

Here $T^{**}\big(J_X(B_X)\big) = J_Y\big(T(B_X)\big) \subseteq J_Y(M)$ and since the latter is weak* closed in Y^{**} we get

$$T^{**}(B_{X^{**}}) \subseteq \overline{T^{**}(J_X(B_X))}^{w^*} = \overline{J_Y(M)}^{w^*} = J_Y(M).$$

Consequently, $\operatorname{Ran} T^{**} = \bigcup_{n\in\mathbb{N}} nT^{**}(B_X) \subseteq J_Y(Y)$.

Conversely, suppose that $\operatorname{Ran} T^{**} \subseteq J_Y(Y)$. Then $T^{**}(B_{X^{**}}) = J_Y(M)$ for some $M \subseteq Y$. Now $B_{X^{**}}$ is weak*-compact by the Banach-Alaoglu Theorem and T^{**} is weak*-continuous, so $T^{**}(B_{X^{**}})$ is weak*-compact. Since J_Y is a weak-to-weak*-homeomorphic embedding it follows that M must be weakly compact and in particular weakly closed. Now

$$J_Y(T(B_X)) = T^{**}(J_X(B_X)) \subseteq J_Y(M),$$

and hence $T(B_X) \subseteq M$. Since M is weakly compact so is the weak closure of $T(B_X)$. Thus T is weakly compact.

Remark: In particular, if Y is reflexive then every bounded linear operator from X to Y is weakly compact.

(b) Suppose first that T is weakly compact. Then by part (a) we have $\operatorname{Ran} T^{**} \subseteq J_Y(Y)$ and hence T^* is weak*-to-weak-continuous. Since B_{Y^*} is weak*-compact by the Banach-Alaoglu Theorem, its image $T^*(B_{Y^*})$ under T^* is weakly compact and in particular weakly closed. Hence T^* is weakly compact.

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Conversely, if T^* is weakly compact then by part (a) we have $\operatorname{Ran} T^{***} \subseteq J_{X^*}(X^*)$, so T^{**} is weak*-to-weak-continuous. If Y is complete then so is $J_Y(Y)$, and therefore it is norm-closed. By convexity it is also weakly closed. Using Goldstine's Theorem we see that

$$T^{**}(B_{X^{**}}) = T^{**}\Big(\overline{J_X(B_X)}^{\mathrm{w}^*}\Big) \subseteq \overline{T^{**}(J_X(B_X))}^{\mathrm{w}^*} = \overline{J_Y(T(B_X))}^{\mathrm{w}} \subseteq J_Y(Y),$$

and hence T is weakly compact by part (a).

Remark: This result is due to V. Gantmacher (1940) and M. Nakamura (1951). It can be viewed as an analogue of Schauder's Theorem for weakly compact operators. Together with part (a) the result implies in particular that every bounded linear operator from X to Y is weakly compact also if X^* is reflexive and Y is complete.

4. Let X, Y be Banach spaces and suppose that $T \in \mathcal{B}(X, Y)$. Show that T is Fredholm if and only if T^* is and that, if both operators are Fredholm, then $\operatorname{ind} T + \operatorname{ind} T^* = 0$.

Solution: If T is Fredholm then $\operatorname{Ran} T$ is closed, and by the Closed Range Theorem so is $\operatorname{Ran} T^*$. Thus $\operatorname{Ran} T^* = (\operatorname{Ker} T)^\circ$, which has finite codimension in X^*. Moreover $\operatorname{Ker} T^* = (\operatorname{Ran} T)^\circ \cong (Y/\operatorname{Ran} T)^*$, which is finite-dimensional. Thus T^* is Fredholm. Conversely, if T^* is Fredholm then $\operatorname{Ran} T^*$ is closed and the Closed Range Theorem implies that $\operatorname{Ran} T = (\operatorname{Ker} T^*)_\circ$, which has finite codimension in Y. Moreover $\operatorname{Ran} T^* = (\operatorname{Ker} T)^\circ$, so $(\operatorname{Ker} T)^* \cong X^*/\operatorname{Ran} T^*$ is finite-dimensional and hence so is $\operatorname{Ker} T$. The above arguments also show that if both T and T^* are Fredholm then $\dim \operatorname{Ker} T^* = \dim Y/\operatorname{Ran} T$ and $\dim \operatorname{Ker} T = \dim X^*/\operatorname{Ran} T^*$, so $\operatorname{ind} T + \operatorname{ind} T^* = 0$.

5. Let X, Y and Z be Banach spaces and let $S \in \mathcal{B}(Y, Z)$ and $T \in \mathcal{B}(X, Y)$.
   (a) Show that if S, T are both Fredholm then so is ST and $\operatorname{ind} ST = \operatorname{ind} S + \operatorname{ind} T$.
   (b) Suppose now that ST is Fredholm. Prove that S is Fredholm if and only if T is Fredholm. Give an example in which neither S nor T is Fredholm.
   (c) Show that if $X = Y = Z$ and $ST = TS$ then ST is Fredholm if and only if S and T are both Fredholm.

Solution:

(a) If S and T are Fredholm then there exist closed finite-codimensional subspaces U and V of X and Y, respectively, such that T maps U isomorphically onto $\operatorname{Ran} T$ and S maps V isomorphically onto $\operatorname{Ran} S$. Thus $\dim X/U = \dim \operatorname{Ker} T$ and $\dim Y/V = \dim \operatorname{Ker} S$. Let $W = V \cap \operatorname{Ran} T$, a closed finite-codimensional subspace of Y. Then the space $M = U \cap T^{-1}(W)$ is closed in X and satisfies $\dim U/M = \dim \operatorname{Ran} T/W$, so M has finite codimension in U and hence in X. Similarly, the space $N = S(W)$ is closed in Z and $\dim \operatorname{Ran} S/N = \dim V/W$, so N has finite codimension in $\operatorname{Ran} S$ and hence in Z. Since ST maps M isomorphically onto N we see that ST is Fredholm, and furthermore

$$\begin{aligned}
\operatorname{ind} ST &= \dim X/M - \dim Z/N \\
&= \dim X/U + \dim \operatorname{Ran} T/W - \dim Z/\operatorname{Ran} S - \dim V/W \\
&= \dim \operatorname{Ker} T - \dim Y/\operatorname{Ran} T + \dim Y/W \\
&\qquad - \dim Z/\operatorname{Ran} S + \dim Y/V - \dim Y/W \\
&= \operatorname{ind} T + \operatorname{ind} S.
\end{aligned}$$

(b) Suppose that ST is Fredholm. Note that $\operatorname{Ker} T \subseteq \operatorname{Ker} ST$ and $\operatorname{Ran} ST \subseteq \operatorname{Ran} S$. Hence $\operatorname{Ker} T$ is finite-dimensional and $\operatorname{Ran} S$ has finite-codimension in Z, so T is Fredholm if and only if $\operatorname{Ran} T$ has finite codimension in Y and S is Fredholm if and only if $\operatorname{Ker} S$ is finite-dimensional. Since ST is Fredholm there exists a closed finite-codimensional subspace U of X such that ST maps U isomorphically onto $\operatorname{Ran} ST$. Let $V = T(U)$. Then $V \cap \operatorname{Ker} S = \{0\}$ and $\operatorname{Ran} T$ is of finite codimension in Y if and only if V is. Since $\operatorname{Ran} ST = S(V)$ has finite codimension in Z there exists a finite-dimensional subspace W of Y such that $Y = \operatorname{Ker} S \oplus V \oplus W$. Hence $\dim Y/V < \infty$ if and only if $\dim \operatorname{Ker} S < \infty$, so T is Fredholm if and only if S is.

For a suitable example let $X = Y = Z = \ell^2$ and consider $Sx = (x_2, x_4, x_6, \dots)$ and $Tx = (0, x_1, 0, x_2, 0, \dots)$. Then ST is the identity operator, and in particular Fredholm, but $\operatorname{Ker} S$ is infinite-dimensional and $\operatorname{Ran} T$ has infinite codimension, so neither S nor T is Fredholm.

(c) We know from part (a) that if S and T are both Fredholm then so is ST. Suppose $Q = ST = TS$ is Fredholm. Then $\operatorname{Ker} T \subseteq \operatorname{Ker} Q$ and $\operatorname{Ran} Q \subseteq \operatorname{Ran} T$, and similarly $\operatorname{Ker} S \subseteq \operatorname{Ker} Q$ and $\operatorname{Ran} Q \subseteq \operatorname{Ran} S$. So S, T are both Fredholm.

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Section C

1. Let X be the complex Banach space ℓ^1 and consider the left-shift operator $T \in \mathcal{B}(X)$ given by $Tx = (x_{n+1})_{n\geq 1}$ for $x = (x_n)_{n\geq 1} \in X$. Moreover let $\Gamma = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$.

   (a) Show that for $\lambda \in \mathbb{C}$ the operator $T - \lambda$ is Fredholm if and only if $\lambda \notin \Gamma$, and determine the index $\operatorname{ind}(T - \lambda)$ whenever it is defined.

   (b) Let p be a complex polynomial. Prove that $p(T)$ is Fredholm if and only if $p^{-1}(\{0\}) \cap \Gamma = \emptyset$ and that, if this condition is satisfied, then

$$\operatorname{ind} p(T) = \frac{1}{2\pi i} \oint_\Gamma \frac{p'(\lambda)}{p(\lambda)} \,\mathrm{d}\,\lambda.$$

Solution:

(a) Since $\|T\| = 1$ we know that for $|\lambda| > 1$ the operator $T - \lambda$ is invertible and hence Fredholm with $\operatorname{ind}(T - \lambda) = 0$. If $|\lambda| < 1$ then it is easy to verify that $\operatorname{Ker}(T - \lambda) = \operatorname{Span}\{(1, \lambda, \lambda^2, \dots)\}$. We now show that $T - \lambda$ is surjective for $|\lambda| < 1$. This can be done directly, or alternatively by noting that if we identify X^* with ℓ^∞ then T^* is the right shift defined by $T^*x = (0, x_1, x_2, \dots)$ for $x = (x_n) \in \ell^\infty$. In particular, T^* is an isometry and $\|T^*x - \lambda x\| \geq (1 - |\lambda|)\|x\|$ for all $x \in \ell^\infty$, so $T^* - \lambda$ is an isomorphic embedding. Since X is complete it follows from a result in lectures that $T - \lambda$ is a quotient operator and in particular surjective. Hence $T - \lambda$ is Fredholm and $\operatorname{ind}(T - \lambda) = 1$ for $|\lambda| < 1$. Since the index is locally constant on the set of Fredholm operators, $T - \lambda$ cannot be Fredholm when $|\lambda| = 1$.

(b) Let p be a complex polynomial. If p is constant then $p(T)$ is Fredholm if and only if p is non-zero. If p is non-constant then we may write it in the form $p(\lambda) = c(\lambda - \lambda_1) \cdots (\lambda - \lambda_n)$ for some $n \geq 1$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ and $c \in \mathbb{C} \setminus \{0\}$. Then

$$p(T) = c \prod_{k=1}^{n} (T - \lambda_k),$$

and by a previous result and induction we obtain that $p(T)$ is Fredholm if and only if $T - \lambda_k$ is Fredholm for $1 \leq k \leq n$. By part (a) we know that $T - \lambda_k$ is Fredholm if and only if $|\lambda_k| \neq 1$, so $p(T)$ is Fredholm if and only if $\{\lambda_1, \dots, \lambda_n\} \cap \Gamma = \emptyset$. But $\{\lambda_1, \dots, \lambda_n\} = p^{-1}(\{0\})$. Thus in all cases $p(T)$ is Fredholm if and only if $p^{-1}(\{0\}) \cap \Gamma = \emptyset$. Moreover, if $p(T)$ is Fredholm then by Q.2 and the first part

$$\operatorname{ind} p(T) = \sum_{k=1}^{n} \operatorname{ind}(T - \lambda_k) = \frac{1}{2\pi i} \sum_{k=1}^{n} \oint_\Gamma \frac{\mathrm{d}\lambda}{\lambda - \lambda_k} = \frac{1}{2\pi i} \oint_\Gamma \frac{p'(\lambda)}{p(\lambda)} \,\mathrm{d}\lambda.$$

Remark: Suppose that p has no zeros on the unit circle Γ, so that $p(T)$ is Fredholm. Let $\Gamma_p = p(\Gamma)$ and parameterise Γ_p (with the orientation inherited from Γ being traversed counterclockwise) by $t \mapsto r(t)e^{i\theta(t)}$, where $r(t) > 0$ and $\theta(t)$ is real for $0 \le t \le 1$. Then $r(0) = r(1)$ and hence

$$\operatorname{ind} p(T) = \frac{1}{2\pi i}\oint_{\Gamma_p} \frac{\mathrm{d}\lambda}{\lambda} = \frac{1}{2\pi i}\int_0^1 \left(\frac{r'(t)}{r(t)} + i\theta'(t)\right)\mathrm{d}t = \frac{\theta(1)-\theta(0)}{2\pi}.$$

Thus $\operatorname{ind} p(T)$ coincides with the *winding number* of the curve Γ_p, that is to say its total number of counterclockwise revolutions about the origin. There are many other situations in which the Fredholm index has a topological interpretation. For instance, the famous *Atiyah-Singer Index Theorem* states that for elliptic differential operators on compact manifolds the Fredholm index equals a certain topological index.

2. Let X be a Banach space and let $\{x_n : n \ge 1\}$ be a Schauder basis for X with basis projections P_n, $n \ge 1$, and let

$$|||x||| = \sup\{\|P_n x\| : n \ge 1\}, \quad x \in X.$$

Prove that $|||\cdot|||$ defines a complete norm on X.

Solution: If $|||x||| = 0$, then $\|P_n x\| = 0$ for all $n \in \mathbb{N}$ and therefore the representation of x in terms of the Schauder basis is $x = \sum_n 0 \cdot x_n$. Hence $x = 0$. Next,

$$|||\lambda x||| = \sup_{n\in\mathbb{N}}\{\|P_n(\lambda x)\|\} = \sup_{n\in\mathbb{N}}\{|\lambda|\cdot\|P_n x\|\} = |\lambda|\cdot\sup_{n\in\mathbb{N}}\{\|Px_n\|\} = |\lambda|\cdot|||x|||$$

for all $\lambda \in \mathbb{F}$. For the triangle inequality, we have

$$\begin{aligned} |||x+y||| &= \sup_{n\in\mathbb{N}}\{\|P_n(x+y)\|\} \\ &= \sup_{n\in\mathbb{N}}\{\|P_n x + P_n y\|\} \\ &\leqslant \sup_{n\in\mathbb{N}}\{\|P_n x\| + \|P_n y\|\} \\ &\leqslant \sup_{n\in\mathbb{N}}\{\|P_n x\|\} + \sup_{n\in\mathbb{N}}\{\|P_n y\|\} \\ &= |||x||| + |||y||| \end{aligned}$$

for all $x, y \in X$.

Now we verify that $|||\cdot|||$ is complete. Let $(a_i)_{i\in\mathbb{N}} = (\alpha_1^i x_1 + \alpha_2^i x_2 + \cdots)_{i\in\mathbb{N}}$ be a Cauchy sequence in $(X, |||\cdot|||)$. Let $\varepsilon > 0$. Then there is some $N \in \mathbb{N}$ such that for all $i, j > N$ we have

$$\begin{aligned} \varepsilon > |||a_i - a_j||| &= \sup_{n\in\mathbb{N}}\left\{\left\|(\alpha_1^i - \alpha_1^j)x_1 + \cdots + (\alpha_n^i - \alpha_n^j)x_n\right\|\right\} \\ &\geqslant \left\|(\alpha_1^i - \alpha_1^j)x_1 + \cdots + (\alpha_n^i - \alpha_n^j)x_n\right\| \end{aligned}$$

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for every $n \in \mathbb{N}$. In particular, this implies that $(\alpha_1^i x_1 + \cdots + \alpha_n^i x_n)_i$ is a Cauchy sequence for each $n \in \mathbb{N}$. It is then easy to show that each $(\alpha_n^i)_i$ is a Cauchy sequence for every $n \in \mathbb{N}$, and therefore that there are constants $\alpha_1, \alpha_2, \cdots$ such that

$$\lim_{i \to \infty} (\alpha_1^i x_1 + \cdots + \alpha_n^i x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n$$

for each $n \in \mathbb{N}$. Formally define $a = \sum_n \alpha_n x_n$; we have not yet shown that the series on the right hand side is norm convergent, but for now we will only be interested in the projections $P_n a = \alpha_1 x_1 + \cdots + \alpha_n x_n$ which are well-defined.

Let $n \in \mathbb{N}$, let $i > N$, and let $j > N$ be such that $\|P_n a - P_n a_j\| < \varepsilon$. Then

$$\|P_n a - P_n a_i\| \leqslant \|P_n a - P_n a_j\| + \|P_n a_j - P_n a_i\| < 2\varepsilon.$$

Since the above inequality is independent of n, we obtain

$$\sup_{n \in \mathbb{N}} \{\|P_n a - P_n a_i\|\} < 2\varepsilon$$

for all $i > N$. It only remains to show that the series defining a is in fact norm convergent so that a is actually a well-defined element of X. To do this, we will show that $(P_n a)_{n \in \mathbb{N}}$ isa Cauchy sequence in $(X, \|\cdot\|)$. Let $m, n > N$. Then for every $i > N$ we have

$$\|P_m a - P_n a\| \leqslant \|P_m a - P_m a_i\| + \|P_m a_i - P_n a_i\| + \|P_n a_i - P_n a\| < 4\varepsilon + \|P_m a_i - P_n a_i\|$$

But $(P_n a_i)$ is Cauchy for every i, so there is some N_i such that $\|P_m a_i - P_n a_i\| < \varepsilon$ for all $m, n > N_i$. So for $m, n > \max\{N, N_i\}$, we obtain $\|P_m a - P_n a\| < 5\varepsilon$, which proves the claim.

3. (a) Prove that if X is a Banach space with a Schauder basis, then every compact operator on X is a norm limit of finite rank operators. [1]

   (b) Show that if $T : X \to Y$ is a finite rank operator, then so too is T^*.

   (c) Suppose that X is a Banach space with a Schauder basis. Show how to use parts (a) and (b) above to deduce that if $T : X \to Y$ is compact, then so too is T^* in this case. Note that this result applies when X is a Hilbert space.

[1] Additional exercise. Show that regardless of separability, every compact operator on a Hilbert space is a limit of finite rank operators.

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