Lie groups. C3.5. HT26 [P. Bousseau]

Lecture 2. [23/01/2026]

Today: left-invt vector fields

"Lie groups are highly symmetric objects"

Def: G Lie group $g \in G$

- Left translation $L_g : G \to G$, $x \mapsto gx$
- Right translation $R_g : G \to G$, $x \mapsto xg$
- Conjugation $C_g : G \to G$, $x \mapsto gxg^{-1} = L_g \circ R_{g^{-1}}(x)$

L_g, R_g differ C_g Lie group isom

$[L_g^{-1} = L_{g^{-1}},\ R_g^{-1} = R_{g^{-1}}]$

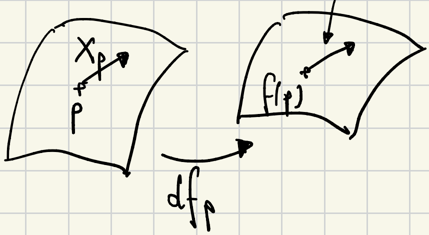
⚠ L_g and R_g are not group homomorphism: $L_g(e) = R_g(e) = g \neq e$ in general.

Recall: a vector field X on a manifold M is a smooth assignement of tangent vector at each point i.e. X smooth section of $\underbrace{TM}_{\text{tangent bundle}}$

Vector space of vector fields $C^\infty(TM)$
(∞-dim if $\dim M > 0$)

- $f : M \to N$ differ $f_* : C^\infty(TM) \to C^\infty(TN)$

defined by $(f_* X)_{f(p)} = df_p(X_p)$



Def: G Lie group.
A vector field X on G is left-invariant if $(L_g)_* X = X$ $\forall g \in G$.

$\mathfrak{g} := \text{Lie}(G) = \{ \text{left-invt vector fields on } G \}$ Lie algebra of G

See next lecture for general def of Lie algebra.

Prop: G Lie group, identity e, dim n.

Then $\mathfrak{g} \xrightarrow{\sim} T_e G$ so is a vector space of dim n.

$X \longmapsto X_e$

Moreover if $(X_1)_e, \dots, (X_n)_e$ is a basis for $T_e G$ and $X_1, \dots, X_n$ are the corresponding left-invt vector fields, then $\forall X \in C^\infty(TG)$, $\exists!\, f_1, \dots, f_n \in C^\infty(G)$ s.t. $X = \sum_{i=1}^{n} f_i X_i$.

Pf: See PS sheet 1. $\square$

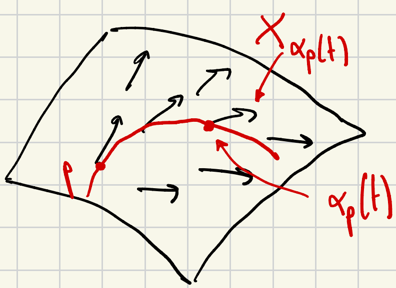
Example: $G = (0, \infty)$ $\times$ coordinate s

General smooth vector field $X = f(s) \frac{d}{ds}$ $f(s)$ smooth function

X left-invariant $\Leftrightarrow$ $f(s) = s f(1)$ $\mathfrak{g} = \text{Span}\langle s \frac{d}{ds} \rangle$

$\forall X \in C^\infty(TM)$ $X = f(s) \frac{d}{ds} = \left(\frac{f(s)}{s}\right)\left(s \frac{d}{ds}\right)$

Recall: M manifold $X \in C^\infty(M)$ $p \in M$ $\rightsquigarrow$ unique integral curve



$\alpha_p : (-\varepsilon, \varepsilon) \to M$

$$\begin{cases} \alpha_p'(t) = X_{\alpha_p(t)} \\ \alpha_p(0) = p \end{cases}$$

→ Flow of X
$\forall p \in X$, $\exists \varepsilon > 0$,
$\exists U$ open, $p \in X$ s.t. $\phi_t : \underset{\substack{\cup \\ p}}{\overset{\text{open}}{U}} \longrightarrow M$ $\forall t \in (-\varepsilon, \varepsilon)$

$q \longmapsto \phi_t(q) = \alpha_q(t)$ exists

$$\frac{\partial \phi_t}{\partial t}\Big|_{t=0}(q) = \alpha_q'(0) = X_{\alpha_q(0)} = X_q \quad \text{so} \quad \boxed{\frac{\partial \phi_t}{\partial t}\Big|_{t=0} = X}$$

<u>Example</u>: $G = (0, \infty)$ $X = s\frac{d}{ds}$ left-invt

$$\begin{cases} \alpha_s'(t) = X_{\alpha_s(t)} = \alpha_s(t) \\ \alpha_s(0) = s \end{cases} \Rightarrow \alpha_s(t) = e^t s$$

defined $\forall s \in G$ $\forall t \in \mathbb{R}$

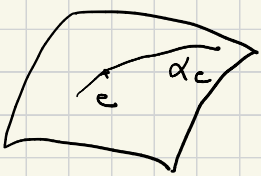
$\Rightarrow$ General fact: $e^{t_1 + t_2} = e^{t_1} e^{t_2}$
$\alpha_e : \mathbb{R} \to (0, \infty)$ group hom

<u>Thm</u>: G Lie group with identity e.

a) $X \in \mathfrak{g}$. Let α_x be the integral curve of X through $x \in G$.

Then: i) $\alpha_x(t)$ is defined $\forall t \in \mathbb{R}$

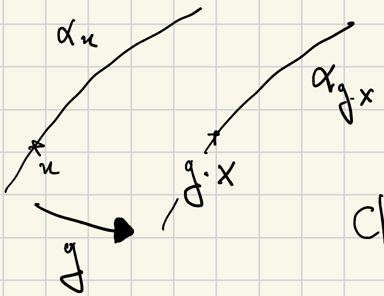
ii) $\alpha_e : \mathbb{R} \to G$ is a Lie group hom



$$\alpha_e(t_1 + t_2) = \alpha_e(t_1)\, \alpha_e(t_2)$$

b) If $f : \mathbb{R} \to G$ is a Lie group hom, $\exists X \in \mathfrak{g}$ s.t. $f = \alpha_e$.

Proof: a) i) Claim: $g \cdot \alpha_x = \alpha_{g \cdot x}$



Proof:

$$g \cdot \alpha_x(0) = g \cdot x = \alpha_{g \cdot x}(0)$$

$$\frac{d}{dt}\left(g \cdot \alpha_x(t)\right) = (dL_g)_{g \cdot \alpha_x(t)}\left(\alpha_x'(t)\right)$$

Chain Rule

$$= (dL_g)_{g \cdot \alpha_x(t)}\left(X_{\alpha_x(t)}\right)$$

$$= \left((L_g)_* X\right)_{g \cdot \alpha_x(t)}$$

$$= X_{g \cdot \alpha_x(t)}$$

$(L_g)_* X = X$

Uniqueness of integral curves

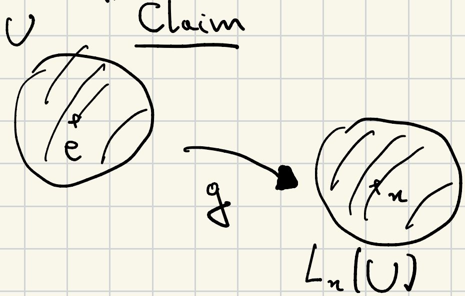
$$\Rightarrow g \cdot \alpha_x = \alpha_{g \cdot x}.$$

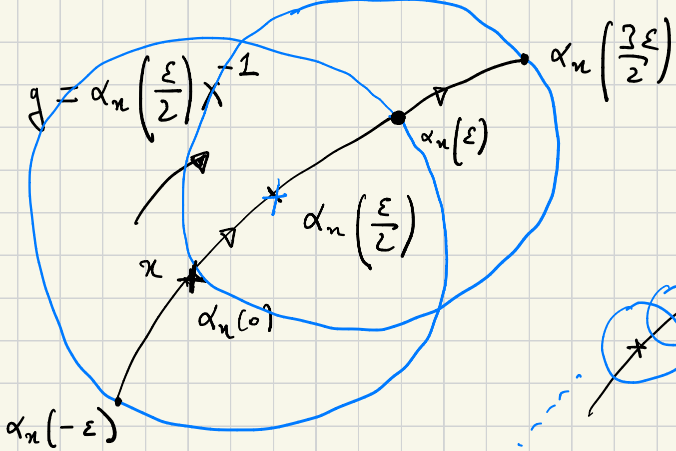
$\square$

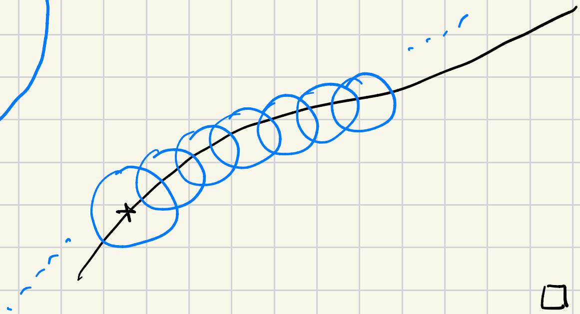
$e \in U$ open set on which we have existence & uniqueness of integral curves of X defined for $t \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Then $\forall x \in G$ $L_x(U)$ open set s.t. .... $\forall t \in (-\varepsilon, \varepsilon)$ (same ε!)

Claim







$\square$

a) ii) <u>Goal</u>: $\alpha_e(t_1+t_2) = \underbrace{\alpha_e(t_1)\,\alpha_e(t_2)}_{}$

$= L_{\alpha_e(t_1)}(\alpha_e(t_2))$ by i)

Both integral curves of X as function of t_2

Fix t_1

At $t_2 = 0$: Both $= \alpha_e(t_1)$

Uniqueness of integral curves $\Rightarrow$ Result

$\square$

b) $f: \mathbb{R} \to G$ smooth group hom $\quad X_e := f'(0) \quad X \in \mathfrak{g}$ corresponding left-invt vector field

$f(t_1+t_2) = f(t_1)\,f(t_2)$

Fix t_1, differentiate with respect to t_2 at $t_2 = 0$:

$$f'(t_1) = dL_{f(t_1)}(X_e)$$
$$= X_{f(t_1)}$$

X left-invt

so $f(t) = \alpha_e(t) \quad \forall t.$

$\square$

Next time: Lie algebras.