Lie groups. C3.5. HT26 [P. Bousseau]

Lecture 3. [26/01/2026]

Today: Lie algebra $\quad X, Y$ vector fields on a manifold M

operators acting on $\vec{\text{functions}}$ $[X,Y] := X \circ Y - Y \circ X = \mathcal{L}_X Y$

If ϕ_t is the flow of X near $p \in M$: $[X,Y]_p = \lim_{t \to 0} \frac{(\phi_{-t})_*(Y_{\phi_t(p)}) - Y_p}{t}$

$$X = \sum_i X_i \frac{\partial}{\partial x_i} \qquad Y = \sum_i Y_i \frac{\partial}{\partial x_i}$$

$$[X,Y] = \sum_{i,j} X_i \frac{\partial}{\partial x_i}\left(Y_j \frac{\partial}{\partial x_j}\right) - Y_i \frac{\partial}{\partial x_i}\left(X_j \frac{\partial}{\partial x_j}\right)$$

$$= \sum_{i,j} \left(X_i \frac{\partial Y_j}{\partial x_i} - Y_i \frac{\partial X_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

$$\phi_t(p) = p + t\vec{X} + \dots$$

$$Y_{\phi_t(p),j} = Y_j(p + t\vec{X} + \dots) = Y_{j,p} + t \sum_i \frac{\partial Y_j}{\partial x_i} X_i + \dots$$

$$(\phi_{-t})_*(Y_{\phi_t(p)})_j = Y_{\phi_t(p),j} - t \sum_i \frac{\partial X_j}{\partial x_i} Y_{\phi_t(p),i} + \dots$$

$$= Y_{j,p} + t \sum_i \frac{\partial Y_j}{\partial x_i} X_i - t \sum_i \frac{\partial X_j}{\partial x_i} Y_i + \dots$$

If $f: M \to N$ differ

then $f_*[X,Y] = [f_*X, f_*Y]$

(see Pb sheet 1)

Lemma: $X, Y \in \mathfrak{g} \Rightarrow [X,Y] \in \mathfrak{g}$

Proof: $(L_g)_*[X,Y] = [(L_g)_* X, (L_g)_* Y] = [X,Y]$. $\square$

$\forall g \in G$

Lie algebra \ $[\,,\,]$: bilinear, skew symmetric & satisfies Jacobi identity:

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Prop: $\mathfrak{g}$ is a Lie algebra. "The Lie algebra of G".

Example: $G = GL(n,\mathbb{R})$ $\quad X, Y \in \mathfrak{g}$

$A \in M(n,\mathbb{R})$ $\quad \alpha_A(t)$ integral curve of X starting at A:

$$\begin{cases} \alpha_A'(t) = X_{\alpha_A(t)} \underset{\substack{\uparrow \\ X \text{ left invt}}}{=} \alpha_A(t) X_I \\ \alpha_A(0) = A \end{cases} \qquad \alpha_A(t) = A e^{tX_I} = \phi_t(A)$$

$$[X,Y]_I = \lim_{t\to 0} \frac{(\phi_{-t})_* (Y_{\phi_t(I)}) - Y_I}{t}$$

$$(\phi_{-t})_*(Y_{\phi_t(I)}) \underset{\substack{\uparrow \\ Y \text{ left-invt}}}{=} (\phi_{-t})_*(\phi_t(I) Y_I) = \phi_t(I) Y_I \phi_{-t}(I)$$

$$= e^{tX_I} Y_I e^{-tX_I}$$

$$= Y_I + t(X_I Y_I - Y_I X_I) + O(t^2)$$

so $[X,Y]_I = [X_I, Y_I]$ $\quad$ Lie bracket = Matrix commutator.

Same $\forall$ Lie group $\subset GL(n,\mathbb{R})$.

Next: representation

Recall: V vector space $\rightarrow$ $\underset{\text{automorphism group of } V}{\mathrm{Aut}(V)} = \{T : V \to V \text{ linear invertible}\}$

E.g.: $V = \mathbb{R}^n$
$\mathrm{Aut}(V) \simeq GL(n, \mathbb{R})$

Def: A representation ρ of G on a vector space V (finite dim real) is a lie group homomorphism $\rho : G \to \mathrm{Aut}(V)$

Ex: $G = SO(n)$ $V = \mathbb{R}^n$ $A \in SO(n)$ $\rho(A)(v) = Av$ $\forall v \in \mathbb{R}^n$

Always have a representation of G on $\mathfrak{g}$:

Def: The adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is:

$$\mathrm{Ad}(g) = (C_g)_* \qquad \forall g \in G$$
$$= (R_{g^{-1}})_* \circ (L_g)_*$$
$$= (R_{g^{-1}})_* \qquad \text{since } (L_g)_* = \mathrm{Id} \text{ on } \mathfrak{g}$$

$C_g(x) = g x g^{-1}$

Lemma: $\forall g \in G$ $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$ is a lie algebra homomorphism:

$$\mathrm{Ad}(g)[X, Y] = [\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y] \quad \forall X, Y \in \mathfrak{g}$$

Proof:

$$(C_g)_*[X, Y] = [(C_g)_* X, (C_g)_* Y] \quad \square$$

Recall: $V \rightsquigarrow \mathrm{End}(V) = \{T : V \to V \text{ linear}\}$ $\quad V \simeq \mathbb{R}^n$ $\mathrm{End}(V) \simeq M(n, \mathbb{R})$

$$= T_I GL(n, \mathbb{R})$$

Def: $\mathrm{ad} = d\mathrm{Ad}_e : \underset{\mathfrak{g}}{\overset{}{T_e G}} \to \mathrm{End}(\mathfrak{g})$ $\quad$ ($T_e G \cong \mathfrak{g}$)

<u>Fact:</u> $\mathrm{ad}(X)(Y) = [X, Y] \quad \forall X, Y \in \mathfrak{g}$

<u>Ex:</u> $G = GL(n, \mathbb{R}) \quad \mathrm{Ad}(A)(B) = ABA^{-1} \quad \forall A \in GL(n, \mathbb{R}), B \in M(n, \mathbb{R})$

$\mathrm{ad}(B)(C) = [B, C] = BC - CB \quad \forall B, C \in M(n, \mathbb{R})$

<u>Prop:</u> Let $f: G \to H$ be a Lie group hom

Then $df_e : T_e G \simeq \mathfrak{g} \longrightarrow T_e G \simeq \mathfrak{g}$ is a Lie algebra hom.

<u>Proof</u> (only sketch/Idea) $\quad X, Y \in \mathfrak{g} \underset{\exists}{\Rightarrow} Z, W \in \mathfrak{h}$ s.t. $Z_e = df_e(X_e)$

Claim: $[Z, W]_e = df_e [X, Y]_e \qquad W_e = df_e(Y_e)$

Idea: α_e integral curve of $X \Rightarrow f \circ \alpha_e$ integral curve of Z. $\square$

<u>Cor:</u> $\mathrm{ad}([X, Y]) = [\mathrm{ad}(X), \mathrm{ad}(Y)]$

$$= \mathrm{ad}(X)\,\mathrm{ad}(Y) - \mathrm{ad}(Y)\,\mathrm{ad}(X).$$

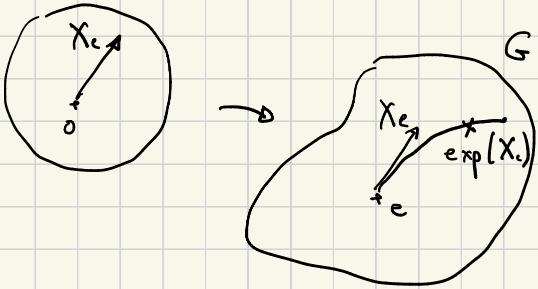
<u>Proof:</u> Take $f = \mathrm{Ad}$ in Proposition. $\square$

<u>Remark:</u> Corollary applied to $Z \Rightarrow$ Jacobi identity.

<u>Def:</u> The <u>exponential map</u> $\exp : T_e G \simeq \mathfrak{g} \to G$ is given by

$$\exp(X_e) := \alpha_e(1) \quad \text{where } \alpha_e = \text{integral curve of } X \in \mathfrak{g} \text{ with given } X_e \in T_e G.$$

$\mathfrak{g} \simeq T_e G$



Ex: $G = S^1$, $\mathfrak{g} = \mathbb{R}$, $\exp(t) = e^{it}$ smooth, surjective but not injective.

Prop: $d\exp_0 : \mathfrak{g} \to \mathfrak{g}$ is the identity map, and so exp is a local diffeomorphism at 0 (inverse function theorem).

Proof: $X \in \mathfrak{g}$ $s \in \mathbb{R}$ $\Rightarrow$ integral curve α_e^s of sX satisfies:

$$\alpha_e^s(t) = \alpha_e(st) \quad \forall t \in \mathbb{R}$$

$$\exp(sX_e) = \alpha_e^s(1) = \alpha_e(s)$$

Differentiate and set $s=0$: $\underbrace{\frac{d}{ds}\exp(sX_e)\Big|_{s=0}}_{= d\exp_0(X_e)} = \alpha_e'(0) = X_e$ $\square$

Ex: $\mathfrak{g}$ connected $\Rightarrow$ $\exp(\mathfrak{g})$ connected

$\exp : M(n,\mathbb{R}) \to GL(n,\mathbb{R})$

$B \mapsto e^B$

not surjective as $GL(n,\mathbb{R})$ disconnected

Rem: exp surjective if G compact & connected.

Ex: $G = SL(2,\mathbb{R})$ $\quad e^B = \left(e^{B/2}\right)^2$

$B \in \mathfrak{sl}(2,\mathbb{R}) = \{B \in M(2,\mathbb{R}) \mid \mathrm{Tr}\,B = 0\}$

but $\begin{pmatrix} -2 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \in SL(2,\mathbb{R})$ so $\notin \mathrm{Im}\exp$

$\neq A^2 \quad \forall A \in SL(2,\mathbb{R})$

$\Rightarrow$ exp not surjective even if $SL(2,\mathbb{R})$ connected (but non-compact).