Lie groups. C3.5. HT 26 [P. Boutteon]

Lecture 4. [30/01/2026]

G Lie group. Recall $\exp: \mathfrak{g} \to G$ $\qquad \exp(X_e) = \alpha_e(1)$ (integral curve of X)

Lem: Let $f: G \to H$ be a Lie group hom. Then

$$\exp(df_e(X_e)) = f(\exp(X_e)).$$

Proof: α_e integral curve of X starting at e.

Then $f \circ \alpha_e$ = integral curve of Z with $Z_e = df_e(X_e)$ | See last time

$$\exp(df_e(X_e)) = f \circ \alpha_e(1) = f(\exp(X_e)). \qquad \square$$

Prop: G connected Lie group. $\forall V_{\ni e}$ open, $\forall g \in G$, $\exists v_1, \dots, v_k \in V$

s.t. $g = v_1^{\pm} \cdots v_k^{\pm}$.

Proof:

Let $F = \{v_1^{\pm} \cdots v_k^{\pm}, v_1, \dots, v_k \in V, k \in \mathbb{N}\}$

$e \in F$ so $F \neq \emptyset$ Enough to show F open and closed.

Open: let $g \in F$ $\qquad \underset{\ni g}{L_g(V)}$ open $\subset F$ ($g \cdot v$, $v \in V$) ✓

Closed: Let $g \in \overline{F}$. $\underset{\ni g}{L_g(V)}$ open containing g so $L_g(V) \cap F \neq \emptyset$

$\exists h \in L_g(V) \cap F$

$h = gv$ $\qquad v \in V$, $h \in F$ | $\Rightarrow g = hv^{-1} \in F$ $\qquad \square$

Prop: Let $f_1, f_2 : G \to H$ be Lie group hom with G connected and $(df_1)_e = (df_2)_e$. Then $f_1 = f_2$.

Proof: $\exists \underset{\substack{\cup \\ 0}}{U}$ open $\subset T_eG$ $\quad \exists \underset{\substack{\cup \\ e}}{V}$ open $\in G$ s.t. $\exp : U \to V$ diffeo

since exp local diffeo at 0 (see last time).

Then $\forall X_e \in U$, $f_1(\exp(X_e)) \underset{\text{Lemma}}{=} \exp((df_1)_e(X_e)) = \exp((df_2)_e(X_e)) \underset{\text{Lemma}}{=} f_2(\exp(X_e))$

so $f_1|_V = f_2|_V$

$\left.\begin{array}{l}\text{G connected} \\ \text{Previous Prop}\end{array}\right] \Rightarrow$ every $g \in G$, $g = v_1^{\pm} \cdots v_k^{\pm}$ $\quad v_i \in V$

$+ f_1, f_2$ group hom $\Rightarrow f_1 = f_2$. $\square$

Remark: $\mathfrak{g} \simeq \mathfrak{h} \not\Rightarrow G = H$ even if G and H connected

Example: $G = SU(2)$, $H = SO(3) \Rightarrow \mathfrak{g} \simeq \mathfrak{h}$ (see Problem Sheet)

but G simply connected whereas H is not ($\pi_1(H) = \mathbb{Z}/2\mathbb{Z}$).

Thm: Let G connected Lie group. The following are equivalent:

a) $\exp : \mathfrak{g} \to G$ is a group homomorphism.

b) G is abelian

c) $G \simeq T^k \times \mathbb{R}^{n-k}$ $\quad n = \dim G$

Lie group isom

Classification of connected abelian Lie groups.

Proof: a) $\Rightarrow$ b) $\exists e \in V$ open in G s.t $\forall v \in V, \exists X \in \mathfrak{g},\ v = \exp(X)$

exp local $\Uparrow$ differ at 0

$v = \exp(X) \Rightarrow \exp^{-1} = \exp(-X)$

$\exp(Y+X) = \exp(Y)\exp(X)$

a): $\forall X, Y \in \mathfrak{g} \quad \exp(X+Y) = \exp(X)\exp(Y)$

G connected $\forall g \in G \quad g = \exp(\pm X_1) \cdots \exp(\pm X_k)$

Prop $\Rightarrow$ $= \exp(\pm X_1 \pm \cdots \pm X_k)$ so G abelian.

b) $\Rightarrow$ a) let $m: G \times G \to G$ be multiplication map.

$(g_1, g_2) \mapsto g_1 g_2$

Note: $dm_e(X,Y) = X + Y$

G abelian

$m(g_1h_1, g_2h_2) = g_1h_1g_2h_2 = g_1g_2h_1h_2 = m(g_1,g_2)\,m(h_1,h_2)$

$\forall g_1, g_2, h_1, h_2 \in G$ — Multiplication on $G \times G$

so $m: G \times G \to G$ is a lie group hom.

so $m(\exp(X), \exp(Y)) = \exp(X)\exp(Y)$

lemma $\to$ ||

$\exp(dm_{(e,e)}(X,Y)) = \exp(X+Y)$.

Note:

a) $\Leftrightarrow$ b) show that $\exp: \mathfrak{g} \to G$ is a surjective lie group hom.

c) $\Rightarrow$ b) Obvious.

b) $\Rightarrow$ c) G abelian and we know $\forall g \in G\ \exists X \in \mathfrak{g},\ g = \exp(X)$.

Note: $\exp(X+Y) = \exp(X)\exp(Y)$ (since b) $\Rightarrow$ a)) $(*)$

so $K := \operatorname{Ker} \exp$ is an additive subgroup of $\mathfrak{g}$.

Then $G \simeq \mathfrak{g}/K$ (1^{st} isom thm) $\qquad \mathfrak{g} \simeq \mathbb{R}^n$

$k := \dim \operatorname{Span}(K)$

Let $X_1, \dots, X_k \in K$ linearly independent.

Consider $F := \{a_1 X_1 + \dots + a_k X_k \in K,\ a_1, \dots, a_n \in [0,1]\}$

F closed bounded in $\mathbb{R}^n$ so F compact.

$(*) \Rightarrow \exp$ local differ at all $X \in \mathfrak{g}$

In particular, $\forall X \in F$, $\exists U$ open s.t. $U_X \cap K = \{X\}$

$[\exp(X) = e] \qquad U_X \ni X$

$\{(U_X), X \in F\}$ open cover of $F \Rightarrow F$ finite.

so $\mathbb{Z}X_1 \oplus \dots \oplus \mathbb{Z}X_k \subset K$ has finite index m

Lagrange theorem $\Rightarrow$

$$m K \subset \mathbb{Z}X_1 \oplus \dots \oplus \mathbb{Z}X_k$$

$$K \subset \underbrace{\frac{1}{m}\mathbb{Z}X_1 \oplus \dots \oplus \mathbb{Z}X_k}_{\simeq \mathbb{Z}^k}$$

K subgroup of a free finitely generated abelian group

Algebra $\Rightarrow$ K is a free abelian group:

$\exists\, Y_1, \dots, Y_k \in K$ linearly independent

s.t.: $K = \{a_1 Y_1 + \dots + a_k Y_k \mid a_i \in \mathbb{Z}\}$

Extend $Y_1, \dots, Y_k$ to a basis $Y_1, \dots, Y_n$ of $\mathfrak{g}$.

Define $f : \mathfrak{g} \longrightarrow T^k \times \mathbb{R}^{n-k}$

$$f(a_1 Y_1 + \dots + a_n Y_n) = \left(e^{2i\pi a_1}, \dots, e^{2i\pi a_k}, a_{k+1}, \dots, a_n\right)$$

$\mathrm{Ker}\, f = K$ so $\mathfrak{g}/K \simeq T^k \times \mathbb{R}^{n-k}$. $\square$

Remark:

$\exp(X+Y) \neq \exp(X)\exp(Y)$ in general.

(if G non-abelian).