B1.2 Set Theory

Sheet 2 — HT26

On this sheet, assume the axioms ZF1-7 (Extensionality, Empty Set, Pairing, Union, Power Set, Comprehension, Infinity).

Section A

1. (a) Let X and Y be sets. Prove that there is a set whose elements are precisely the *surjective* functions $X \to Y$.

   (b) Let X be a set. Prove that there is a set consisting precisely of all strict total orders on X.

2. For each of the following, find an equivalent $\mathcal{L}$-formula using no abbreviations (so using only the symbols $=, \in, \neg, \wedge, \vee, \rightarrow, \leftrightarrow$ $\forall, \exists$, and variables, and not using the notation $\forall x \in y$).

   (a) $\forall x \in y \; x \subseteq y$.

   (b) $\bigcup x = \bigcup y$.

   (c) $x \subseteq \{z \in \mathcal{P}(y) : \emptyset \in z\}$

3. Prove that addition on $\mathbb{N}$ is commutative by proving that for all $n, m \in \mathbb{N}$:

   (a) $0 + n = n$;

   (b) $m + n^+ = m^+ + n$;

   (c) $m + n = n + m$.

Martin Bays: bays@maths.ox.ac.uk

Section B

4. Let a be a set. Show that $\{a\} \times \{a\} = \{\{\{a\}\}\}$.

5. Prove the following:

   (a) Any subset of a finite set is finite.

   [*Hint: First show by induction that any subset of a natural number is finite.*]

   (b) If X is finite and $f : X \to X$ is an injective function, then f is surjective.

   [*Hint: if $f : n^+ \to n^+$ is a non-surjective injection, compose with a transposition to obtain an injection $f' : n^+ \to n$, then consider $f'|_n$.*]

6. Prove that multiplication on $\mathbb{N}$ is commutative by proving that the following hold for all $n, m \in \mathbb{N}$. You may use the associativity and commutativity of addition (proven in lectures and Question 3 respectively).

   (a) $0 \cdot n = 0$;

   (b) $m^+ \cdot n = m \cdot n + n$;

   (c) $m \cdot n = n \cdot m$.

7. Prove the existence of the factorial function: show that a function $\mathbb{N} \to \mathbb{N}$ exists such that, writing it as $n \mapsto n!$, we have $0! = 1$, and $(n^+)! = n^+ \cdot n!$ for all $n \in \mathbb{N}$.

   [*Hint: use definition by recursion indirectly.*]

8. A *Peano system* is a triple (A, s, a_0) in which A is a set, $a_0 \in A$, and $s : A \to A$ is a function which

   - is injective,
   - does not have a_0 in its range, and
   - satisfies the Principle of Induction: if $S \subseteq A$, $a_0 \in S$ and $\forall a(a \in S \to s(a) \in S)$, then $S = A$.

   (a) Prove that $(\mathbb{N}, x \mapsto x^+, 0)$ is a Peano system. [*You may refer to results proven in lectures.*]

   (b) Suppose (A, s, a_0) is a Peano system. Prove that there exists an isomorphism from $(\mathbb{N}, x \mapsto x^+, 0)$ to (A, s, a_0), that is, there is a bijection $f : \mathbb{N} \to A$ such that $f(0) = a_0$, and $f(n^+) = s(f(n))$ for all $n \in \mathbb{N}$.

   [*Hence, up to isomorphism, $(\mathbb{N}, x \mapsto x^+, 0)$ is the unique Peano system.*]

Martin Bays: bays@maths.ox.ac.uk

9. Prove that the following properties of a set X are equivalent:

   (i) There exists an injective function $f : \mathbb{N} \to X$.

   (ii) There exists a function $g : X \to X$ which is injective but not surjective.

   [*Hint: For (ii)⇒(i), define f by recursion.*]

Martin Bays: bays@maths.ox.ac.uk

Section C

10. A set X is *amorphous* if it is infinite, but $X \setminus Y$ is finite for any infinite $Y \subseteq X$. It is consistent with ZF (but not with ZFC) that amorphous sets exist.

    Show that if X is amorphous, then the pigeonhole principle holds for X: any injective function $f : X \to X$ is surjective.

    Hence, under ZF, the pigeonhole principle does not characterise finiteness.

Martin Bays: bays@maths.ox.ac.uk