Lie groups. C3.5 HT26 [P. Boalch]

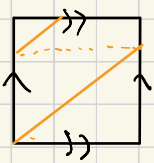
Lecture 5 [02/02/2026] Today: Lie subgroups.

Def: G Lie group. A <u>Lie subgroup</u> of G is a subgroup H of G s.t. H is a Lie group and the inclusion $i : H \hookrightarrow G$ is a Lie group homomorphism.

⚠ Lie group topology on H might not be the induced topology from G.

Example: $\alpha \in \mathbb{R}$ Consider $f_\alpha : \mathbb{R} \to T^2$

$$t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$$

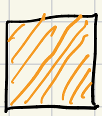


α = slope

Curve in T^2 : $\underbrace{\text{might close up}}_{\alpha \in \mathbb{Q}}$ $\underbrace{\text{or not}}_{\alpha \in \mathbb{R}\setminus\mathbb{Q}}$

$\alpha \in \mathbb{R}\setminus\mathbb{Q}$ $\quad f_\alpha : \mathbb{R} \to T^2$

injective, subgroup, Lie group homomorphism
but does not have induced topology. [$\forall\, 0 \in U$ open, $\forall N \in \mathbb{N}, \exists n > N$ s.t. $f_\alpha(n) \in U$]



Recall: $f : M \to N$ smooth map between manifolds. <u>Immersion</u> if df_p injective $\forall p \in M$. <u>Embedding</u> if $f : M \xrightarrow{\sim} f(M)$ homeomorphism.

Ex: $\forall \alpha \in \mathbb{R}\setminus\mathbb{Q}$, f_α is an injective immersion but not an embedding.

Def: If $i : H \to G$ is a Lie subgroup. We say that it is embedded if i is an embedding.

Example: $SO(n)$ is an embedded Lie subgroup of $O(n)$.

More generally: $\forall G$, $G_0 \hookrightarrow G$ is an embedded Lie subgroup.
↳ connected component of e

Recall: $\underset{\dim m}{M} \subset \underset{\dim n}{N}$ is an embedded submanifold $\Leftrightarrow$ $\forall p \in M$, $\exists \underset{\ni p}{U}$ open,
$\exists F : U \to \mathbb{R}^{n-m}$ smooth map s.t.
$F(p) = 0$, 0 regular value of F, $F^{-1}(0) \cap U = M \cap U$.

Kind of inverse of the regular value theorem.

Topological Consequence: M is locally closed in N:
$\forall p \in M$, $\exists \underset{\ni p}{U}$ open s.t. $M \cap U = \overline{M} \cap U$.

Thm: let H be a subgroup of a Lie group G. Then:
H is an embedded Lie subgroup of G $\Leftrightarrow$ H is closed.

Proof: $\boxed{\Rightarrow}$ $y \in \overline{H}$ Goal: $y \in H$

H is locally closed in G: $\exists \underset{\ni e}{U}$ open s.t. $H \cap U = \overline{H} \cap U$.

let $U^{-1} := \{u^{-1}, u \in U\} \ni e$
open

Consider $L_y(U^{-1})$ open so $L_y(U^{-1}) \cap H = \emptyset$
$\forall y \quad (y \in \overline{H})$

$\exists x \in L_y(U^{-1}) \cap H$

$x = y u^{-1} \quad u \in U$

$x^{-1} y = u \in U$

$x \in H \quad y \in \overline{H}$

$\Rightarrow x^{-1} y \in \overline{H}$ so $x^{-1} y \in \overline{H} \cap U = H \cap U$

so $y \in H$.

$\boxed{\Leftarrow}$ Not easy.

[Claim: It is enough to show that H is an embedded Lie subgroup near e (Then use left translations).

exp local diffeo at 0: $\exists \underset{0}{V} \subset \mathfrak{g} = T_e G$ open, $\exists \underset{e}{U} \subset G$ open s.t. $\exp : V \to U$ diffeo.

Let $\log : U \to V$ be the inverse of exp.

Define $S = \{ X \in T_e G \mid \exp(tX) \in H, \forall t \in \mathbb{R} \}$

<u>Goal</u>: S is a vector space and "$S = T_e H$"

Set up for later: identify $T_e G \simeq \mathbb{R}^n$, $\langle \cdot , \cdot \rangle$ inner product, $\| \cdot \|$ norm.

Is S a vector subspace of $\mathfrak{g} \simeq T_e G$?

- $0 \in S$ since $\exp(0) = e \in H$
- $X \in S$, $\lambda \in \mathbb{R} \Rightarrow \lambda X \in S$ $\left[\exp(t\lambda X) = \exp(t' X) \quad t' = t\lambda \right]$

. $X, Y \in T_e G$ Define $\forall s$ sufficiently small s.t. $\exp(sX)\exp(sY) \in U$

$$\gamma(s) := \log(\exp(sX)\exp(sY)) \in V \subset T_e G$$

$d\exp_0 = id$
so $d\log_e = id$ $\Big| \Rightarrow \gamma'(0) = X + Y$ so

Chain rule + product rule

$$\exp(sX)\exp(sY)$$
$$= \exp(s(X+Y) + \mathcal{O}(s^2))$$ by Taylor's Theorem

Fix $t \in \mathbb{R}$, consider $n >> 0$

Now $X, Y \in S$: $\underbrace{\left(\exp\left(\frac{t}{n}X\right)\exp\left(\frac{t}{n}Y\right)\right)^n}_{\in H} = \exp\left(\frac{t}{n}(X+Y) + \mathcal{O}\left(\frac{t^2}{n^2}\right)\right)^n$

$$= \exp\left(t(X+Y) + \mathcal{O}\left(\frac{t^2}{n}\right)\right)$$
$$\xrightarrow[n \to +\infty]{} \exp(t(X+Y))$$

so $\exp(t(X+Y)) \in \overline{H} = H$

$\forall t \in \mathbb{R} \Rightarrow X + Y \in S$

so S is a vector space.

Use $\langle \cdot, \cdot \rangle$: $T_e G = S \oplus S^\perp$

$$\varphi : T_e G = S \oplus S^\perp \longrightarrow G$$
$$(X, Z) \longmapsto \exp(X)\exp(Z)$$

φ is a smooth map. $\varphi(S) \subset H$ Converse? Near e

Contradiction argument. Suppose $\exists (X_n, Z_n) \in S \oplus S^\perp$

s.t. $\|(X_n, Z_n)\| \longrightarrow 0$ with $Z_n \neq 0$ $\forall n$ and $\varphi(X_n, Z_n) \in H$ $\forall n$

$$\varphi(X_n, Z_n) = \underbrace{\exp(X_n)}_{\in H} \exp(Z_n)$$

[H subgroup]

so $\exp(Z_n) \in H$ $\forall n$

Consider $\dfrac{Z_n}{\|Z_n\|}$ bounded sequence in unit sphere in $S^\perp$

$\Rightarrow \exists$ convergent subsequence. Still call it Z_n.

$$\frac{Z_n}{\|Z_n\|} \xrightarrow[n\to\infty]{} Y \qquad Y \in S^\perp, \quad \|Y\| = 1$$

Let $t \in \mathbb{R}\setminus\{0\}$, $\dfrac{|t|}{\|Z_n\|} \xrightarrow[n\to\infty]{} \infty$ $\quad m_n := \left\lfloor \dfrac{t}{\|Z_n\|} \right\rfloor \in \mathbb{Z}$ $\quad m_n \|Z_n\| \xrightarrow[n\to\infty]{} t$

$m_n \in \mathbb{Z} \Rightarrow \exp(m_n Z_n) = (\exp(Z_n))^{m_n} \in H$ [H subgroup]

$$\exp(m_n Z_n) = \exp\left(m_n \|Z_n\| \frac{Z_n}{\|Z_n\|}\right) \xrightarrow[n\to\infty]{} \exp(tY) \in \overline{H} = H \quad [H \text{ closed}]$$

$\forall t \in \mathbb{R}$ so $Y \in S$ so $Y \in S \cap S^\perp = \{0\}$

$\Rightarrow Y = 0$ contradiction with $\|Y\| = 1$.

So: $\exists\, 0 \in \tilde{V} \subset V \subset T_eG$, $\exists\, e \in \tilde{U} \subset U \subset G$ s.t. $\tilde{U} = \exp(\tilde{V})$

$$\begin{cases} \varphi(X, Z) \in H \\ (X, Z) \in \tilde{V} \end{cases} \iff Z = 0.$$

so $H \cap \tilde{U} = F^{-1}(0) \cap \tilde{U}$ where $F(\varphi(X,Z)) = Z$.

$F : \tilde{U} \to S^{\perp}$ $\qquad \Rightarrow H$ submanifold.

$dF_e = \mathrm{Id}$

$\square$

<u>Example:</u> G Lie group. $Z(G)$ center is a closed subgroup.

Thm $\Rightarrow$ $Z(G)$ is an embedded Lie subgroup.

<u>Thm:</u> G, H Lie groups. $f : G \to H$ group homomorphism.

Then f continuous $\Rightarrow$ f smooth $\Rightarrow$ f Lie group hom.

<u>Proof:</u>

Graph of f: $\Gamma = \{(g, f(g)), g \in G\} \underset{\text{subgroup because } f \text{ group hom.}}{\subset} G \times H$

f continuous $\Rightarrow \Gamma$ closed $\underset{\text{Previous Thm}}{\Rightarrow}$ Embedded Lie subgroup

$$\begin{tikzcd}
& \Gamma \arrow[dl, "\pi_G|_\Gamma"'] \arrow[dr, "\pi_H|_\Gamma \text{ smooth}"] \subset G \times H & \\
G & & H
\end{tikzcd}$$

$$f = (\pi_H)_\Gamma \circ (\pi_G|_\Gamma)^{-1} \text{ smooth!}$$

$\square$

Next time: Lie subalgebras.