A12204H1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.2
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.2
Master of Science in Mathematical Sciences: Paper C5.2

Elasticity and Plasticity

TRINITY TERM 2025

Monday 02 June, 9:30am to 11:15am

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. In the absence of body forces, the Navier equation for the displacement $\mathbf{u}(\mathbf{x},t)$ in a linear elastic material is given by

$$\rho\frac{\partial^2\mathbf{u}}{\partial t^2}=(\lambda+\mu)\boldsymbol{\nabla}\left(\boldsymbol{\nabla}\cdot\mathbf{u}\right)+\mu\boldsymbol{\nabla}^2\mathbf{u},$$

where ρ is the density and the Lamé constants λ and μ satisfy $\lambda+\frac{2}{3}\mu>0$ and $\mu>0$.

(a) [10 marks] Consider nontrivial harmonic travelling wave solutions of the form

$$\mathbf{u}=\mathbf{a}\exp\big(\mathrm{i}\left(\mathbf{k}\cdot\mathbf{x}-\omega t\right)\big),$$

where the complex amplitude $\mathbf{a}$, wavenumber vector $\mathbf{k}\neq 0$ and frequency ω are constant. Show that there exists a scalar A and a vector $\mathbf{B}$ such that $\mathbf{a}=A\mathbf{k}+\mathbf{B}\wedge\mathbf{k}$ with $\mathbf{B}\cdot\mathbf{k}=0$. Are A and $\mathbf{B}$ unique? Justify your answer.

Deduce that either (i) $\mathbf{B}=\mathbf{0}$ and $\omega^2=c_{\mathrm{p}}^2\,\mathbf{k}\cdot\mathbf{k}$ or (ii) $A=0$ and $\omega^2=c_{\mathrm{s}}^2\,\mathbf{k}\cdot\mathbf{k}$, where the wave speeds c_{p} and c_{s} should be defined. Show that $c_{\mathrm{s}}<c_{\mathrm{p}}$ and comment on the relationship between the direction of propagation and the displacement in both cases.

(b) [15 marks] A linear elastic material undergoes plane strain in $y<0$ below an elastic membrane of surface density σ that is stretched to a uniform tension T in the plane $y=0$. The membrane performs small purely transverse displacements in the y-direction that are independent of z, so that the displacement $\mathbf{u}=(u(x,y,t),v(x,y,t),0)^T$ satisfies

$$u=0,\qquad \sigma\frac{\partial^2 v}{\partial t^2}-T\frac{\partial^2 v}{\partial x^2}+(\lambda+2\mu)\frac{\partial v}{\partial y}=0\qquad\text{on}\qquad y=0.$$

A *Rayleigh* wave propagates in the x-direction with speed $c>0$ and wavenumber $k>0$, while decaying exponentially as $y\to-\infty$, the in-plane displacement being

$$\begin{pmatrix}u\\ v\end{pmatrix}=a\begin{pmatrix}\mathrm{i}k\\ \kappa_{\mathrm{p}}\end{pmatrix}\exp\left(\mathrm{i}k(x-ct)+\kappa_{\mathrm{p}}y\right)+b\begin{pmatrix}\mathrm{i}\kappa_{\mathrm{s}}\\ k\end{pmatrix}\exp\left(\mathrm{i}k(x-ct)+\kappa_{\mathrm{s}}y\right),\qquad(\star)$$

where a, b, κ_{p} and κ_{s} are constants.

(i) Show that a nontrivial solution of the form $(\star)$ can only exist if c is less than c_{s} and related to k by

$$\frac{c_{\mathrm{s}}}{\sqrt{c_{\mathrm{s}}^2-c^2}}-\frac{\sqrt{c_{\mathrm{p}}^2-c^2}}{c_{\mathrm{p}}}=\frac{\rho}{\sigma k}\frac{c^2}{(c^2-c_{\mathrm{m}}^2)},$$

where the membrane wave speed $c_{\mathrm{m}}=\sqrt{T/\sigma}$.

(ii) Show that if $c_{\mathrm{m}}<c_{\mathrm{s}}$ then there is a unique solution for c.

2. An elastic string at $z = w(x)$ is stretched to a tension $T(x)$ under a smooth shallow rigid obstacle at $z = f(x)$ in the absence of any body forces, where the positive z-direction points upwards. The ends of the string are fixed so that the small transverse displacement $w(x)$ satisfies $w(-L) = 0$ and $w(L) = 0$. You may assume that T, w and $\mathrm{d}w/\mathrm{d}x$ are all continuous at points where the string makes or loses contact with the obstacle

(a) [11 marks] (i) By balancing forces on a small segment of the string, show that $T(x)$ and $w(x)$ satisfy

$$\frac{\mathrm{d}T}{\mathrm{d}x} = 0, \qquad T\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = 0$$

whenever the string is not in contact with the obstacle. How are these equations modified when the string is in contact with the obstacle?

(ii) Hence write down the *linear complementarity problem* satisfied by $w(x)$ explaining the physical interpretations of any inequalities that arise.

(iii) Let $\mathcal{V} = \{v \in C^1[-L, L] : v(\pm L) = 0,\ v \leqslant f\}$ and for $v \in \mathcal{V}$ define the functional

$$U[v] = \int_{-L}^{L} \frac{T}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 \mathrm{d}x.$$

Noting that $w \in \mathcal{V}$, show that if $v \in \mathcal{V}$ then

$$\int_{-L}^{L} \left(\frac{\mathrm{d}v}{\mathrm{d}x} - \frac{\mathrm{d}w}{\mathrm{d}x}\right) T\frac{\mathrm{d}w}{\mathrm{d}x}\,\mathrm{d}x = \int_{-L}^{L} (f - v)T\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\,\mathrm{d}x.$$

Hence show that $U[w] \leqslant U[v]$ for all $v \in \mathcal{V}$ and interpret this result physically.

(b) [8 marks] Suppose that the obstacle is given by

$$f(x) = -\delta + \frac{\kappa}{2}(x - a)^2,$$

where the parameters δ, κ and a satisfy $0 \leqslant \delta \leqslant \min\{\kappa(\pm L - a)^2/2\}$, $\kappa > 0$ and $|a| < L$.

(i) Show that if $a = 0$ then the string makes contact with the obstacle in a region $-s \leqslant x \leqslant s$, where

$$s = L - \sqrt{L^2 - \frac{2\delta}{\kappa}}.$$

(ii) Hence, or otherwise, show that if $a \neq 0$, then the string makes contact with the obstacle in a region $-s_- \leqslant x \leqslant s_+$, where $s_\pm$ should be determined.

(c) [6 marks] The system described in part (b) is used as a catapult to launch the obstacle which has mass M. The quantities T, L, κ and a are held fixed throughout the shot for which $a = 0$. The obstacle is pushed down to its furthest extent so that $\delta = \kappa L^2/2$ at time $t = 0$ and then released from rest so that δ decreases with t until contact is lost. The effects of gravity and the inertia of the string are negligible.

Write down Newton's second law for the obstacle and deduce that its kinetic energy at the instant that it loses contact with the string is equal to the elastic energy initially stored in the string.

3. A perfectly plastic material with yield stress $\tau_{\rm Y}$ undergoes quasi-steady radially symmetric strain in the annulus $a < r < b$. The displacement field is given by $\mathbf{u} = u(r,t)\mathbf{e}_r$, where (r,θ) denote plane polar coordinates, t denotes time and the unit vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$ point in the r- and θ-direction, respectively. The stress tensor is diagonal with entries τ_{rr} and $\tau_{\theta\theta}$ that satisfy the Cauchy equation

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0.$$

The radial displacement $u = Vt$ is imposed on the inner boundary at $r = a$, where V is a positive constant, while $u = 0$ on $r = b$.

(a) [5 marks] Evaluate the shear stress on an infinitesimal line element with unit normal $\mathbf{n} = \mathbf{e}_r \cos\alpha + \mathbf{e}_\theta \sin\alpha$, and hence show that the maximum shear stress over all inclination angles α is given by the *Tresca yield function*

$$f = \frac{1}{2}|\tau_{rr} - \tau_{\theta\theta}|.$$

State the condition under which the material is elastic and the equation satisfied by f when it is plastic.

(b) [8 marks] First supposing that the material remains elastic, show that the displacement satisfies the differential equation

$$\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = 0,$$

and hence evaluate the displacement in $a < r < b$. Deduce that as t increases from zero yield first occurs at $r = a$ when t reaches the critical value

$$t_{\rm c1} = \left(1 - \frac{a^2}{b^2}\right)\frac{a\tau_{\rm Y}}{2\mu V}.$$

[*You should assume the linear elastic constitutive relations*

$$\tau_{rr} = (\lambda + 2\mu)\frac{\partial u}{\partial r} + \lambda\frac{u}{r}, \qquad \tau_{\theta\theta} = \lambda\frac{\partial u}{\partial r} + (\lambda + 2\mu)\frac{u}{r},$$

where λ and μ are the Lamé constants.]

(c) [12 marks] Show that as t increases beyond $t_{\rm c1}$, the material is plastic in the annular region $a < r < s(t)$, where the location $r = s(t)$ of the elastic/plastic free boundary should be determined. Deduce that the whole annulus yields when t reaches the second critical value

$$t_{\rm c2} = \left(\frac{a^2 + b^2}{2a^2}\right)t_{\rm c1}.$$

[*You should assume that the associated flow rule in the plastic region is given by*

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) = \Lambda\frac{\partial f}{\partial \tau_{rr}}, \qquad \frac{\partial}{\partial t}\left(\frac{u}{r}\right) = \Lambda\frac{\partial f}{\partial \tau_{\theta\theta}},$$

where $\Lambda(r,t)$ is a Lagrange multiplier, and that the dilatation, $\partial u/\partial r + u/r$, is continuous at the elastic/plastic free boundary.]

C5.2/2025/Q1

(a) $\mathbf{k}\wedge(\mathbf{k}\wedge\mathbf{a}) = (\mathbf{k}\cdot\mathbf{a})\mathbf{k} - (\mathbf{k}\cdot\mathbf{k})\mathbf{a} \Rightarrow \mathbf{a} = \frac{\mathbf{k}\cdot\mathbf{a}}{\mathbf{k}\cdot\mathbf{k}}\mathbf{k} + \left(\frac{\mathbf{k}\wedge\mathbf{a}}{\mathbf{k}\cdot\mathbf{k}}\right)\wedge\mathbf{k}$

Hence $\exists A\in\mathbb{R}, \mathbf{B}\in\mathbb{R}^3$ s.t. $\mathbf{a} = A\mathbf{k} + \mathbf{B}\wedge\mathbf{k}$ with $\mathbf{B}\cdot\mathbf{k} = 0$, viz. $A = \frac{\mathbf{k}\cdot\mathbf{a}}{\mathbf{k}\cdot\mathbf{k}}$, $\mathbf{B} = \frac{\mathbf{k}\wedge\mathbf{a}}{\mathbf{k}\cdot\mathbf{k}}$.

For uniqueness suppose $\mathbf{a} = A_1\mathbf{k} + \mathbf{B}_1\wedge\mathbf{k} = A_2\mathbf{k} + \mathbf{B}_2\wedge\mathbf{k}$ with $\mathbf{B}_1\cdot\mathbf{k} = \mathbf{B}_2\cdot\mathbf{k} = 0$, so that $(A_1 - A_2)\mathbf{k} + (\mathbf{B}_1 - \mathbf{B}_2)\wedge\mathbf{k} = \mathbf{0}$ $(+)$ with $(\mathbf{B}_1 - \mathbf{B}_2)\cdot\mathbf{k} = 0$ $(\#)$.

Then $(+)\cdot\mathbf{k} \Rightarrow A_1 = A_2$, while $(+)\wedge\mathbf{k} \Rightarrow \mathbf{B}_1 = \mathbf{B}_2$ using $(\#)$. B3

If $\mathbf{u} = \mathbf{a}e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$, then

$$\frac{\partial^2\mathbf{u}}{\partial t^2} = (-i\omega)^2\mathbf{a}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

$$\operatorname{div}\mathbf{u} = i(\mathbf{k}\cdot\mathbf{a})\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

$$\operatorname{grad}\operatorname{div}\mathbf{u} = -(\mathbf{k}\cdot\mathbf{a})\mathbf{k}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

$$\nabla^2\mathbf{u} = -(\mathbf{k}\cdot\mathbf{k})\mathbf{a}\, e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

so the Navier equation becomes $-\rho\omega^2 = -(\lambda+\mu)(\mathbf{k}\cdot\mathbf{a})\mathbf{k} - \mu(\mathbf{k}\cdot\mathbf{k})\mathbf{a}$

Plug in $\mathbf{a} = A\mathbf{k} + \mathbf{B}\wedge\mathbf{k} \Rightarrow A(\rho\omega^2 - (\lambda+2\mu)\mathbf{k}\cdot\mathbf{k})\mathbf{k} + (\rho\omega^2 - \mu\mathbf{k}\cdot\mathbf{k})\mathbf{B}\wedge\mathbf{k} = \mathbf{0}$

Dot with $\mathbf{k} \Rightarrow A(\rho\omega^2 - (\lambda+2\mu)\mathbf{k}\cdot\mathbf{k}) = 0$

Cross with $\mathbf{k} \Rightarrow \mathbf{B}(\rho\omega^2 - \mu\mathbf{k}\cdot\mathbf{k}) = \mathbf{0}$

For nontrivial $\mathbf{a}$ we must have

either (i) $\mathbf{B} = \mathbf{0} \Rightarrow A \neq 0 \Rightarrow \omega^2 = c_p^2\,\mathbf{k}\cdot\mathbf{k}$ with $c_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$

or (ii) $A = 0 \Rightarrow \mathbf{B} \neq \mathbf{0} \Rightarrow \omega^2 = c_p^2\,\mathbf{k}\cdot\mathbf{k}$ with $c_s = \sqrt{\frac{\mu}{\rho}}$ B5

So $c_p^2 = \frac{\lambda + \frac{2}{3}\mu + \frac{1}{3}\mu + \mu}{\rho} > \frac{\mu}{\rho} = c_s^2 \Rightarrow c_s < c_p$ as $\lambda + \frac{2}{3}\mu, \mu > 0$

In case (i) of a P-wave the direction of propagation $\mathbf{k}/|\mathbf{k}|$ is parallel to the displacement $\mathbf{u}$ as $\mathbf{a} = A\mathbf{k}$, while in case (ii) of an S-wave it is perpendicular.
B2

$\boxed{10}$

(b)(i) Writing the ansatz in the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = ia\begin{pmatrix} k \\ -i\kappa_p \end{pmatrix} \exp\left\{ i\begin{pmatrix} k \\ -i\kappa_p \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} - i\omega t \right\} + b\begin{pmatrix} i\kappa_s \\ k \end{pmatrix} \exp\left\{ i\begin{pmatrix} k \\ -i\kappa_s \end{pmatrix}\cdot\begin{pmatrix} x \\ y \end{pmatrix} - i\omega t \right\},$$

where $\omega = ck$, we see that it is a superposition of a P-wave and an S-wave with wavevectors $\begin{pmatrix} k \\ -i\kappa_p \end{pmatrix}$ and $\begin{pmatrix} k \\ -i\kappa_s \end{pmatrix}$ respectively.

So the dispersion relations for P- and S-waves in (a)(i) and (ii) give

$$c^2 k^2 = \omega^2 = c_p^2 (k^2 - \kappa_p^2) = c_s^2 (k^2 - \kappa_s^2)$$

so that
$$\kappa_p^2 = k^2\left(1 - \frac{c^2}{c_p^2}\right), \quad \kappa_s^2 = k^2\left(1 - \frac{c^2}{c_s^2}\right)$$

For the Rayleigh wave to decay as $y \to -\infty$, we need $\mathrm{Re}(\kappa_p) > 0$ and $\mathrm{Re}(\kappa_s) > 0$, which can only be the case if $c < c_p$ and $c < c_s$, so the propagation speed must satisfy $c < c_s < c_p$, as required. S5

Now plug the displacements

$$u = (iak e^{\kappa_p y} + ib\kappa_s e^{\kappa_s y}) e^{ikx - ikct}$$
$$v = (a\kappa_p e^{\kappa_p y} + bk e^{\kappa_s y}) e^{ikx - ikct}$$

into the BCs on $y = 0$:

$$ak + b\kappa_s = 0$$
$$\big(\underbrace{\sigma(-ikc)^2 - T(ik)^2}_{= -\sigma(c^2 - c_m^2)k^2}\big)(a\kappa_p + bk) + \underbrace{(\lambda + 2\mu)}_{= \rho c_p^2}(a\kappa_p^2 + bk\kappa_s) = 0$$

$$\Rightarrow \begin{pmatrix} k & \kappa_s \\ \rho c_p^2 \kappa_p^2 - \sigma(c^2 - c_m^2)k^2\kappa_p & \rho c_p^2 k\kappa_s - \sigma(c^2 - c_m^2)k^3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Nontrivial solutions $\begin{pmatrix} a \\ b \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Leftrightarrow$ determinant of this matrix is zero, i.e. $\rho c_p^2 k^2 k_s - \sigma(c^2-c_m^2)k^4 - \rho c_p^2 k_p^2 k_s + \sigma(c^2-c_m^2)k^2 k_p k_s = 0$

$\Leftrightarrow \quad \rho c_p^2 k_s(k^2-k_p^2) = \sigma k^2(c^2-c_m^2)(k^2-k_p k_s)$

But $k_p = k\sqrt{1-\frac{c^2}{c_p^2}}$, $k_s = k\sqrt{1-\frac{c^2}{c_s^2}}$ as $k>0$ and $c<c_s<c_p$, so

$$\rho c_p^2 k\sqrt{1-\frac{c^2}{c_s^2}}\left(k^2-k^2+k^2\frac{c^2}{c_p^2}\right) = \sigma k^2(c^2-c_m^2)k^2\left(1-\sqrt{1-\frac{c^2}{c_p^2}}\sqrt{1-\frac{c^2}{c_s^2}}\right)$$

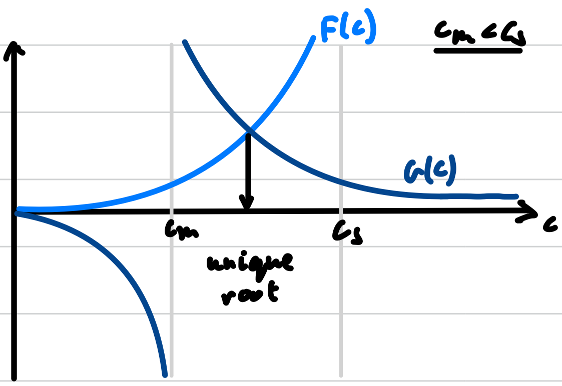
$$\Leftrightarrow \quad \rho k^3 c^2 = \sigma k^4(c^2-c_m^2)\left(\frac{1}{\sqrt{1-\frac{c^2}{c_s^2}}}-\sqrt{1-\frac{c^2}{c_p^2}}\right)$$

$$\Leftrightarrow \quad \underbrace{\frac{c_s}{\sqrt{c_s^2-c^2}}-\frac{\sqrt{c_p^2-c^2}}{c_p}}_{F(c)} = \underbrace{\frac{\rho}{\sigma k}\,\frac{c^2}{c^2-c_m^2}}_{G(c)} \quad \text{as required}$$

S|N6

(b)(ii) Note $F'(c) = \frac{c_s c}{(c_s^2-c^2)^{3/2}} + \frac{c}{c_p(c_p^2-c^2)^{1/2}} > 0$ for $c\in(0,c_s)$ with $F(0)=0$ and $F(c)\to\infty$ as $c\to c_s-$, so $F(c)>0$ and monotonic $\nearrow$ on $(0,c_s)$

Note $G'(c) = -\frac{\rho}{\sigma k}\frac{2c_m^2 c}{(c^2-c_m^2)} < 0$ for $c\in(0,c_m)\cup(c_m,\infty)$, so $G(c)$ is monotonic $\searrow$ on $(0,c_m)$ and on (c_m,∞), with $G(0)=0$, $G(c)\to\pm\infty$ as $c\to c_m\pm$ and $G(c)\to\frac{\rho}{\sigma k}+$ as $c\to\infty$



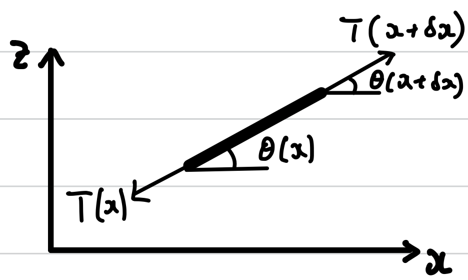
If $c_m<c_s$, then $F(c)>0>\frac{\sigma}{\rho k}G(c)$ for $c\in(0,c_m)$, so no solution for c, while $F(c)\nearrow$ and $G(c)\searrow$ for $c\in(c_m,c_s)$ with $G(c)-F(c)\to+\infty$ as $c\to c_m+$ and $G(c)-F(c)\to-\infty$ as $c\to c_s-$, so that there is a unique root $c\in(c_m,c_s)$, as illustrated.

N4

$\boxed{15}$

C5.2/2025/Q2

(a)(i) When not in contact a force balance on a small segment of the string gives

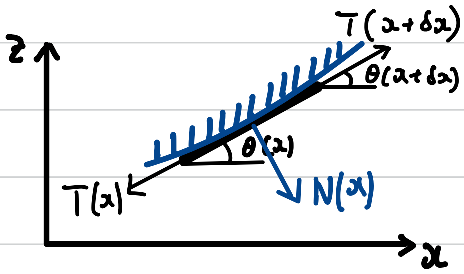


$$\left[T\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right]_x^{x+\delta x} = \begin{pmatrix}0\\0\end{pmatrix} \underset{(\delta x \to 0)}{\Rightarrow} \frac{d}{dx}\left(T\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right) = \begin{pmatrix}0\\0\end{pmatrix}$$

But $|\theta| \ll 1 \Rightarrow \cos\theta \sim 1$, $\sin\theta \sim \theta \sim \tan\theta \sim \frac{\partial w}{\partial x}$

$$\Rightarrow \frac{\partial T}{\partial x} = 0,\ T\frac{\partial^2 w}{\partial x^2} = 0 \text{ to leading order}$$

When in contact, include the normal reaction force $N(x)$ exerted by the obstacle on the string $\Rightarrow$



$$\left[T\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}\right]_x^{x+\delta x} + \int_x^{x+\delta x} N\begin{pmatrix}\sin\theta\\ -\cos\theta\end{pmatrix} dx = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \frac{\partial T}{\partial x} = 0,\ T\frac{\partial^2 w}{\partial x^2} = N \text{ to leading order}$$

B4

(a)(ii) Hence, either (I) $Tw'' = 0$ in regions of non-contact, where the impenetrability of the obstacle $\Rightarrow w \leq f$

or (II) $w = f$ in regions of contact, where the obstacle can "push" but not "pull" on the string $\Rightarrow Tw'' = N \geq 0$

Combine $\Rightarrow$ linear complementarity problem $(w-f)Tw'' = 0$, $w - f \leq 0$, $Tw'' \geq 0$

B3

(a)(iii) So $0 = \int_{-L}^{L} (f - v + v - w)Tw''\,dx = \int_{-L}^{L}(f-v)Tw''\,dx - \int_{-L}^{L}(v-w)'Tw'\,dx$ where integration by parts is OK because w' and v' are continuous and w' piecewise differentiable.

Rearrange to $\int_{-L}^{L}(v' - w')Tw'\,dx = \int_{-L}^{L}(f-v)Tw''\,dx$, as required

Now $U[v]-U[w] = \int_{-L}^{L} \frac{T}{2}\left((v')^2-(w')^2\right)dx$

$$= \int_{-L}^{L} \frac{T}{2}(v'-w')^2 + (v'-w')Tw'\,dx$$

$$= \int_{-L}^{L} \underbrace{\frac{T}{2}(v'-w')^2}_{\geq 0} + \underbrace{(f-v)Tw''}_{\geq 0}\,dx \qquad \text{(from identity above)}$$

≥ 0 as required.

Thus the displacement minimizes the net elastic energy subject to not penetrating the obstacle.

S/B4

$\boxed{11}$

(b)(i) For $a=0$, seek a solution with contact set $|x| < s < 1$.

By symmetry $w(x)$ is even, so we need only solve

① $w''=0$ for $s<x<1$ with $w(s)=f(s)$, $w'(s)=f'(s)$, $w(L)=0$ and $f(x)=-\delta+\frac{\gamma}{2}x^2$

So $w(x)=f(s)+f'(s)(x-s)$ with $f(s)+f'(s)(L-s)=0$

$\Rightarrow -\delta+\frac{\gamma}{2}s^2+\gamma s(L-s)=0$ or $\frac{1}{2}\gamma s^2-\gamma Ls+\delta=0$

$\Rightarrow s = L-\sqrt{L^2-\frac{2\delta}{\gamma}}$, where we chose the $-\sqrt{\;}$ because $s<L$

S4

(b)(ii) If $a \neq 0$ we seek a contact set with $-s_- < x < s_+$, where

② $w''=0$ for $s_+<x<L$ with $w(s_+)=f(s_+)$, $w'(s_+)=f'(s_+)$, $w(L)=0$

③ $w''=0$ for $-L<x<s_-$ with $w(s_-)=f(s_-)$, $w'(s_-)=f'(s_-)$, $w(-L)=0$

wherein $f(x)=-\delta+\frac{\gamma}{2}(x-a)^2$

For ② let $x=a+\hat{x}$ to recover ① with $s\mapsto s_+-a$, $L\mapsto L-a$, so that

$s_+-a = L-a-\sqrt{(L-a)^2-\frac{2\delta}{\gamma}}$, i.e. $s_+ = L-\sqrt{(L-a)^2-\frac{2\delta}{\gamma}}$

Similarly for ③ let $x = a - \hat{x}$ to recover ① with $s \mapsto a + s_-$, $L \mapsto L + a$, so that $a + s_- = L + a - \sqrt{(L+a)^2 - \frac{2\delta}{\kappa}}$, i.e. $s_- = L - \sqrt{(L+a)^2 - \frac{2\delta}{\kappa}}$ S4

$\boxed{8}$

(c) Newton's second law for the obstacle is $M\ddot{\delta} = -F(\delta)$, where $F = \int_{-s}^{s} N\,dx$ is the net upward force exerted by the string on the obstacle (by Newton's third law)

Given the intial conditions $\delta = \frac{\kappa L^2}{2}$, $\dot{\delta} = 0$ when $t = 0$ say (so $s = L$)

Lose contact when $s = 0$, i.e. when $\delta = 0$, $\dot{\delta} = V$ when $t = t_{shot}$ say

So $\int_{t=0}^{t=t_{shot}} \text{ODE} \times \dot{\delta} \Rightarrow \frac{1}{2}MV^2 = -\int_0^{t_{shot}} F(\delta)\dot{\delta}\,dt = \int_0^{\kappa L^2/2} F(\delta)\,d\delta$

But $F(\delta) = \int_{-s}^{s} T w''\,dx = 2Tw'(s) = 2Tf'(s) = 2T\kappa s = 2T\kappa\left(L - \sqrt{L^2 - \frac{\kappa\delta}{2}}\right)$

and $\int_0^{\kappa L^2/2} F(\delta)\,d\delta = 2T\kappa\left[L\delta + \frac{\kappa}{3}\left(L^2 - \frac{2\delta}{\kappa}\right)^{\frac{3}{2}}\right]_{\delta=0}^{\delta=\frac{\kappa L^2}{2}} = 2T\kappa\left(\frac{\kappa L^3}{2} - \frac{\kappa L^3}{3}\right) = \frac{1}{3}T\kappa^2 L^3$

Initial elastic energy is $U[w]\big|_{t=0}$ with $w\big|_{t=0} = f = \frac{\kappa}{2}(x^2 - L^2)$ for $|x| < s = L$

$\Rightarrow U[w]\big|_{t=0} = \int_{-L}^{L} \frac{T}{2} w_x^2\big|_{t=0}\,dx = \int_0^L T\kappa^2 x^2\,dx = \frac{1}{3}T\kappa^2 L^3$

Combine $\Rightarrow \frac{1}{2}MV^2 = \frac{1}{3}T\kappa^2 L^3 = U[w]\big|_{t=0}$

Hence the kinetic energy of the obstacle at the instant it loses contact (LHS) is equal to the elastic energy initially stored in the string (RHS).

N6

$\boxed{6}$

C5.2/2025/Q3

(a) The shear stress on an infinitesimal line element with normal $\underline{n} = \underline{e}_r \cos\alpha + \underline{e}_\theta \sin\alpha$ and tangent $\underline{t} = -\underline{e}_r \sin\alpha + \underline{e}_\theta \cos\alpha$ is

$$\sigma = (-\sin\alpha \quad \cos\theta)\begin{pmatrix} \tau_{rr} & 0 \\ 0 & \tau_{\theta\theta} \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = \tfrac{1}{2}(\tau_{\theta\theta} - \tau_{rr})\sin 2\alpha$$

so $f := \max_\alpha |\sigma| = \frac{1}{2}|\tau_{rr} - \tau_{\theta\theta}|$ because $\max_\alpha |\sin 2\alpha| = 1$. B3

If $f < \tau_Y$ then the material is elastic, while $f = \tau_Y$ when the material is plastic. B2

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(b) When the material is elastic, we substitute the constitutive relations into the Cauchy equation to obtain

$$0 = \frac{\partial}{\partial r}\left((\lambda+2\mu)\frac{\partial u}{\partial r} + \lambda\frac{u}{r}\right) + 2\mu\underbrace{\left(\frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2}\right)}_{=\frac{\partial}{\partial r}\left(\frac{u}{r}\right)}$$

$$= \frac{\partial}{\partial r}\left((\lambda+2\mu)\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right)\right)$$

giving $\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = 0$, as required. B2

So $\frac{\partial u}{\partial r} + \frac{u}{r} = 2A \Rightarrow \frac{\partial}{\partial r}(ru) = 2Ar \Rightarrow u = Ar + \frac{B}{r}$ (A, B constants)

Now apply the boundary conditions.

$u = 0$ on $r = b \Rightarrow 0 = Ab + \frac{B}{b} \Rightarrow u = A\left(r - \frac{b^2}{r}\right)$ (+)

$u = Vt$ on $r = a \Rightarrow Vt = A\left(a - \frac{b^2}{a}\right) \Rightarrow A = \frac{aVt}{a^2 - b^2}$

giving $u = \frac{aVt}{b^2 - a^2}\left(\frac{b^2}{r} - r\right)$ for $a < r < b$.

So $\tau_{rr} - \tau_{\theta\theta} = 2\mu\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) = \frac{2\mu a V t}{b^2 - a^2}\left(-\frac{b^2}{r^2} - 1 - \frac{b^2}{r^2} + 1\right)$

giving $f = \frac{1}{2}|\tau_{rr} - \tau_{\theta\theta}| = \frac{2\mu a b^2 V t}{(b^2 - a^2) r^2}$ $\Rightarrow$ f is max when r is min, i.e. at $r = a$.

Hence, as t is increases from 0, yield first occurs at $r = a$ when

$f(a) = \frac{2\mu b^2 V t}{a(b^2 - a^2)} = \tau_Y \Rightarrow t = t_u = \left(1 - \frac{a^2}{b^2}\right)\frac{a\tau_Y}{2\mu V}$, as required.

S6 $\boxed{8}$

(c) For $t > t_u$, material must yield in a neighborhood of $r = a$ by continuity, say $a < r < s < b$, with s TBD.

In $s < r < b$, still have the elastic solution given by (+), which satisfies $u = 0$ on $r = b$.

Now we fix A by imposing the yield condition at $r = s$, which requires $f = \frac{1}{2}(\tau_{\theta\theta} - \tau_{rr}) = \tau_Y$ at $r = s$, with the sign determined by how yield condition was satisfied initially.

So $f = \frac{1}{2}(\tau_{\theta\theta} - \tau_{rr}) = \mu\left(\frac{u}{r} - \frac{\partial u}{\partial r}\right) = \mu A\left(1 - \frac{b^2}{r^2} - 1 - \frac{b^2}{r^2}\right) = -\frac{2\mu b^2 A}{r^2}$,

giving $-\frac{2\mu b^2 A}{s^2} = f(s) = \tau_Y \Rightarrow A = -\frac{s^2 \tau_Y}{2\mu b^2}$

giving $u = \frac{s^2 \tau_Y}{2\mu b^2}\left(\frac{b^2}{r} - r\right)$ for $s < r < b$. (++)

S4

In the plastic region in $a < r < s$, we have $f = \frac{1}{2}(\tau_{\theta\theta} - \tau_{rr}) = \tau_Y$ from the yield condition in part (a), as well as the associated flow rules

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) = \Lambda \frac{\partial}{\partial \tau_{rr}}\left(\frac{1}{2}|\tau_{rr} - \tau_{\theta\theta}|\right) = -\frac{\Lambda}{2}$$

$$\frac{\partial}{\partial t}\left(\frac{u}{r}\right) = \Lambda \frac{\partial}{\partial \tau_{\theta\theta}}\left(\frac{1}{2}|\tau_{rr} - \tau_{\theta\theta}|\right) = +\frac{\Lambda}{2}$$

Summing them $\Rightarrow \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r} + \frac{u}{r}\right) = 0$ for $a < r < s$ S2

Assuming s continues to increase with t for $t > t_{u}$, we deduce that the dilatation $\frac{\partial u}{\partial r} + \frac{u}{r}$ at a point must remain equal to its value when the material first yielded at that point.

But (H) $\Rightarrow \frac{\partial u}{\partial r} + \frac{u}{r} = \frac{s^2 \tau_Y}{2\mu b^2}\left(-\frac{b}{r^2} - 1 + \frac{b}{r^2} - 1\right) = -\frac{s^2 \tau_Y}{\mu b^2}$ for $s < r < b$

$$\Rightarrow \frac{\partial u}{\partial r} + \frac{u}{r} = -\frac{r^2 \tau_Y}{\mu b^2} \quad \text{for } a < r < s \text{ (assuming continuity at } r = s)$$

Hence $\frac{\partial}{\partial r}(ru) = -\frac{r^3 \tau_Y}{\mu b^2}$, giving $u = \frac{c}{r} - \frac{r^3 \tau_Y}{4\mu b^2}$ for $a < r < s$

Apply $u = Vt$ at $r = a \Rightarrow Vt = \frac{c}{a} - \frac{a^3 \tau_Y}{4\mu b^2} \Rightarrow u = Vt\frac{a}{r} + \frac{\tau_Y}{4\mu b^2}\left(\frac{a^4}{r} - r^3\right)$ for $a < r < s$

Then continuity of u at $r = s \Rightarrow \frac{s^2 \tau_Y}{2\mu b^2}\left(\frac{b^2}{s} - s\right) = Vt\frac{a}{s} + \frac{\tau_Y}{4\mu b^2}\left(\frac{a^4}{s} - s^3\right)$

$$\Rightarrow 2s^2(b^2 - s^2) = \frac{4\mu a b^2 Vt}{\tau_Y} + a^4 - s^4$$

$$\Rightarrow s^4 - 2b^2 s^2 + a^4 + \frac{4\mu a b^2 Vt}{\tau_Y} = 0$$

$$\Rightarrow s^2 = b^2 \pm \sqrt{b^4 - a^4 - \frac{4\mu a b^2 Vt}{\tau_Y}}$$

But $s = a$ when $t = t_u$, so we must choose the $-$ sign, giving

$$s = \sqrt{b^2 - \sqrt{b^4 - a^4 - \frac{4\mu a b^2 Vt}{\tau_Y}}} \quad \text{for } t > t_u$$

The whole annulus yields when $s = b \Rightarrow b^4 - a^4 - \frac{4\mu a b^2 Vt}{\tau_Y} = 0$

$\Rightarrow t = t_{u2} = \frac{(b^4 - a^4)\tau_Y}{4\mu a b^2 V} = \left(\frac{a^2 + b^2}{2a^2}\right) t_u$, as required. N6

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