Lie groups. C3.5. HT26 [P. Bousseau]

Lecture 6 [05/02/26]

Today: Lie algebras and their connection to Lie subgroups.

Def: A vector subspace $\mathfrak{h} \subset \mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is a Lie subalgebra if $\forall X, Y \in \mathfrak{h}$, $[X, Y] \in \mathfrak{h}$.

Ex: H Lie subgroup of Lie group G, then $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra.

Thm: Let G be a Lie group. $\exists$ 1:1 correspondence

$\mathfrak{g} = \mathrm{Lie}(G)$

$$F : \left\{ \begin{array}{c} \text{Lie subalgebras} \\ \mathfrak{h} \subset \mathfrak{g} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{connected Lie subgroups} \\ H \subset G \end{array} \right\}$$

s.t. $\quad F(\mathrm{Lie}(H)) = H \quad \forall H \subset G$ connected Lie subgroup.

Proof:

$\mathfrak{h} \subset \mathfrak{g}$ Lie subalgebra, viewed as left-invariant vector fields.

Let $\{Y_1, \dots, Y_m\}$ be a basis of $\mathfrak{h}$.

Define $\quad E = \{ E_g = \mathrm{Span}\{(Y_1)_g, \dots, (Y_m)_g\}, g \in G \}$

E is a smooth family of m-dim subspaces of tangent spaces to G

$\to$ "E is a distribution of rank m"

Section of E: $\displaystyle\sum_{i=1}^{m} a_i Y_i \qquad a_i \in C^\infty(G) \quad i = 1, \dots, m$

Compute:

$$\Big[\sum_i a_i Y_i, \sum_j b_j Y_j\Big] = \sum_{i,j} a_i Y_i(b_j) \overset{\in \mathfrak{h}}{Y_j} - b_j Y_j(a_i) \overset{\in \mathfrak{h}}{Y_i}$$

$$+ a_i b_j \underbrace{[Y_i, Y_j]}_{} \in \mathfrak{h} \text{ because } \mathfrak{h} \text{ Lie subalgebra.}$$

so E is an "integrable distribution".

Frobenius Thm in differential geometry $\Rightarrow$

- $\exists$ integral connected submanifold $f: H \to G$ with $e \in H$
  such that $df_h(T_h H) = E_{f(h)} \subset T_{f(h)} G$.
  (like "integral curves")
- $\exists!$ maximal such H.

Why is H a subgroup? $h \in H$ $\quad L_{h^{-1}} H$ connected integral submanifold of E
$\ni e$

$\Rightarrow L_{h^{-1}} H \subset H$
$\uparrow$
H maximal

$h^{-1} g \in H \ \forall g \in H$ so H is a subgroup.

H Lie group?

$m_H : H \times H \to H$ smooth

$\Leftrightarrow f \circ m_H : H \times H \to G$ smooth

Restriction of multiplication in G to H, so smooth.
Inversion on H also smooth

H Lie subgroup.
(and connected)

$\square$

Global theory.

Ex: G Lie group G_0 connected component of e

Then G_0 normal subgroup and $G/G_0 = \pi_0(G)$ group of connected components.

(see Pb Sheet 1) $\qquad$ countable discrete group

Ex: $p, q \in \mathbb{N}\setminus\{0\}$ s.t. $p+q=n$ $\quad Q=\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$

Define $O(p,q)=\{A\in M_n(\mathbb{R}) \mid {}^{T}AQA=Q\}$

What is $\pi_0(O(p,q))$? $\quad \pi_0(O(p,q))=\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$

$SO(p,q)=\{A\in O(p,q)\mid \det(A)=1\}$

$\pi_0(SO(p,q))=\mathbb{Z}/2\mathbb{Z}$

Recall: A smooth surjective map $\pi: M\to N$ between manifolds is a covering map if $\forall p\in N$, $\exists$ open $U\ni p$ s.t. $\pi^{-1}(U)=\coprod_i V_i$ $\uparrow$ disjoint open in M

and $\pi|_{V_i}: V_i \xrightarrow{\sim} U$ diffeo.

Remark: If M, N compact, π smooth surjective, $d\pi_p$ isom $\forall p$

Then π covering map, and $|\pi^{-1}(p)|=k<\infty$

"k-fold covering"

Ex: $Ad: SU(2)\to \mathrm{Aut}(\mathrm{Lie}(SU(2)))\simeq GL(3,\mathbb{R})$

has image $SO(3)$ and $Ad: SU(2)\to SO(3)$ is a 2-fold cover.

Ex: $\exists$ covering map: $\pi: SL(2,\mathbb{C})\to SO(1,3)_0$

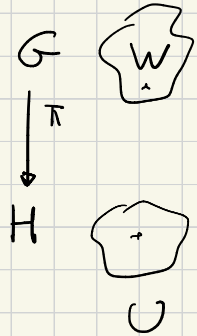
Thm: $\pi: G\to H$ Lie group hom with H connected.

Then π covering map $\Leftrightarrow$ $d\pi_e$ isom

Proof: $\boxed{\Rightarrow}$ Clear.

$\boxed{\Leftarrow}$ $d\pi_e$ isom : $\exists$ open $W \ni e$ in G, open $U \ni e$ in H

s.t. $\pi: W \to U$ diffeo



H connected $\Rightarrow$ H generated by U & U^{-1}

+ π group hom $\Rightarrow$ π surjective.

Consider $f: G \times G \to G$ smooth

$(g_1, g_2) \mapsto g_1 g_2^{-1}$

$\underset{\ni (e,e)}{f^{-1}(W)}$ open in $G \times G$ $\Rightarrow \exists\, V \ni e$ open in G, $V \subset W$

s.t. $V \times V \subset f^{-1}(W)$

$e \in \pi(V) \subset U$ open

If $\pi(g_1) = \pi(g_2) \Leftrightarrow \pi(g_1 g_2^{-1}) = e$

$\Leftrightarrow g_1 g_2^{-1} = k \in \operatorname{Ker} \pi$

$\Leftrightarrow g_1 = k g_2$ for some $k \in \operatorname{Ker} \pi$.

$$\pi^{-1}(\pi(V)) = \bigcup_{k \in \operatorname{Ker}\pi} \underbrace{L_k(V)}_{\text{open}}$$

$\pi : L_k(V) \to \pi(V)$ diffeo

Suffices to show that the $L_k(V)$ are disjoint [then left translation $\Rightarrow$ neighb of every pt]

$k \in \operatorname{Ker}\pi$ $\quad g_1 \in L_k(V) \cap V \neq \emptyset$

$f(g_1, g_2) = g_1 g_2^{-1} = k$ $\quad g_1 \in V \quad g_1 = k g_2 \quad g_2 \in V$

$\pi \circ f(g_1, g_2) = \pi(k) = e$

$g_1, g_2 \in V$

$\Rightarrow g_1 g_2^{-1} \in W$ $\quad \pi|_W$ diffeo $\Rightarrow g_1 g_2^{-1} = e \Rightarrow k = e.$ $\square$

Cor: Let $\pi: G \to H$ be a Lie group covering hom.

Then a) $\ker \pi$ normal discrete subgroup of G.

b) If G is connected, $\ker \pi$ is a subgroup of $Z(G)$.

Pf: a): Immediate from previous proof.

b) $g \in G$ $\quad k \in \ker \pi$ $\quad$ Goal: $kg = gk$

G connected $\exists$ path $g(t)$

$g(0) = e \quad g(1) = g \qquad f(t) := g(t)\, k\, g(t)^{-1}$

$f: [0,1] \to G$

$\ker \pi$ normal $\Rightarrow \pi(f(t)) \in \ker \pi \quad \forall t$

discrete so $\pi(f(t)) = \text{cst} = \pi(f(0)) = k$

$g k g^{-1} = k.$

$\square$