

Lie groups. C3.5. HT26 [P. Bousseau]

1

Lecture 7 [09/02/26]

Def: G connected lie group.

$\exists!$ (up to diffeo) simply connected manifold \tilde{G}

with covering map $\pi: \tilde{G} \rightarrow G$.

\tilde{G} : Universal cover of G .

$\tilde{G} = \{ [\gamma] \text{ homotopy classes of path } \gamma: [0,1] \rightarrow G \text{ s.t. } \gamma(0) = e \}$
with fixed $\gamma(1)$

$$\pi: \tilde{G} \rightarrow G$$

$$[\gamma]t \mapsto \gamma(t)$$

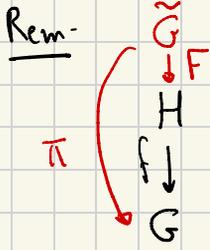
\tilde{G} is a lie group: identity $(\gamma_0(t) = e \forall t)$
• Multiplication $[\gamma_1] \cdot [\gamma_2] = [\gamma_2(t) \gamma_1(t)]$
• Inverse $[\gamma]^{-1} = [\gamma(t)^{-1}]$
 π : lie group hom.

Ex: $G = S^1 = SO(2) = U(1) \Rightarrow \tilde{G} = \mathbb{R}$

Ex: $G = SO(3) \Rightarrow \tilde{G} = SU(2)$

$$n \geq 3$$

$$\tilde{SO}(n) =: Spin(n)$$



G, H connected Lie group hom & covering

$\Rightarrow \exists! F: \tilde{G} \rightarrow H$ s.t. $f \circ F = \pi$

so $H = \tilde{G} / \text{Ker } F$ In particular: $G = \tilde{G} / \text{Ker } \pi$

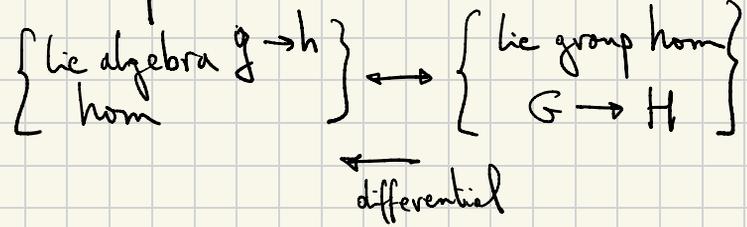
See end of last time. [normal discrete subgroup of $Z(\tilde{G})$]

Ex: $SL(2, \mathbb{R})$ $\text{Ker } \pi = \mathbb{Z}$

does not embed in any $GL(N, \mathbb{R})$ (but any compact Lie group does).

Thm: G, H Lie groups, G simply connected.

(a) \exists 1-1 correspondence

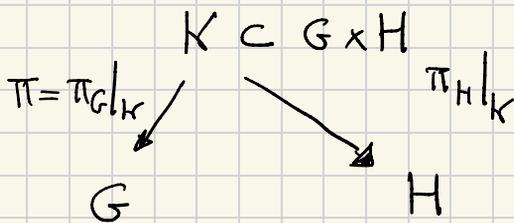


(b) H simply connected, and $\mathfrak{g} \cong \mathfrak{h} \Rightarrow G \cong H$.

Proof: (a) \Rightarrow (b) : Clear.

Proof of (a). $F: \mathfrak{g} \rightarrow \mathfrak{h}$ Lie algebra hom. Let $k = \{(X, F(X)) \mid X \in \mathfrak{g}\}$
 Lie subalgebra (as F hom). $\subset \mathfrak{g} \oplus \mathfrak{h}$
 \parallel
 $\text{Lie}(G \times H)$

Thm last time $\Rightarrow \exists!$ connected Lie group K of $G \times H$ s.t. $T_e K = k$



G simply connected

π lie group hom & $d\pi_{(e,c)}$ isom $\Rightarrow \pi$ covering $\Rightarrow \pi$ differ

↑
Thm last time

let $f = \pi_{H/K} \circ \pi^{-1}$: lie group hom with $df_e = F$.

G connected $\Rightarrow f$ unique
(generated by neighb of identity)

□

Without proof:

Thm [Ado's Theorem] of finite-dimensional lie algebra ($/\mathbb{R}$)

$\Rightarrow \exists$ injective lie alg hom $\mathfrak{g} \rightarrow \mathfrak{gl}(N, \mathbb{R}) = M_N(\mathbb{R})$ for some N

($\Rightarrow \exists$ connected lie group $G \subset GL(N, \mathbb{R})$ s.t. $\mathfrak{g} = \text{Lie}(G)$).

Thm [lie 3rd Theorem] \exists 1-1 correspondence:

$\left\{ \begin{array}{l} \text{lie algebras} \\ \text{(finite dim } / \mathbb{R}) \end{array} \right\} / \text{isom} \leftrightarrow \left\{ \begin{array}{l} \text{Simply connected} \\ \text{lie groups} \end{array} \right\} / \text{isom}.$

Representation theory

Goal: Representation theory of compact lie groups, very similar to representation theory of finite groups.

Def: A representation of a Lie group G on a vector space V is a Lie group hom $\rho: G \rightarrow \text{Aut}(V)$ (finite-dim)
 [Note: enough to check ρ continuous]

Ex: ρ representation of G on V . Action of G on functions on V :
 $\phi(g)f = f \circ \rho(g^{-1}) = f \circ \rho(g)^{-1}$

Ex: U_n representation of $U(1)$ on \mathbb{C} , $\forall n \in \mathbb{Z}$:

$$\rho_n(e^{i\theta}) z = e^{in\theta} z$$

$U(1) \quad \mathbb{C}$

Ex: V_n $n \in \mathbb{N}$ representations of $SU(2)$

$$\mathbb{C}^{n+1} = \text{Span} \{ z_1^n, z_1^{n-1} z_2, \dots, z_2^n \} \quad z_1, z_2 \in \mathbb{C}$$

$$\rho(A)f = f \circ A^{-1} \quad f = f(z_1, z_2)$$

$SU(2)$

Rem: ρ representation of G on $V \Rightarrow \sigma = d\rho_e: \mathfrak{g} \rightarrow \text{End}(V)$ Lie algebra representation

Ex: $U_n \quad G = U(1) \quad \mathfrak{g} = i\mathbb{R}$

$$\sigma(i\theta) z = \left. \frac{d}{dt} \right|_{t \neq 0} \rho_n(e^{it\theta}) z = \left. \frac{d}{dt} \right|_{t \neq 0} e^{int\theta} z = in\theta z$$

Lemma: G simply connected. $\forall \sigma: \mathfrak{g} \rightarrow \text{End}(V)$ Lie algebra rep,

$\exists \rho: G \rightarrow \text{Aut}(V)$ rep of G on V s.t. $d\rho_e = \sigma$.

Proof: Previous Thm Today. \square

Ex: V_n representation of $SO(3) \iff n$ even.

Since $SO(3) = SU(2)/\{\pm I\}$ and $\rho(-I) = \rho(I) \iff n$ even.

Def: ρ_1, ρ_2 : rep of G on V_1, V_2 are isomorphic if $\exists T: V_1 \xrightarrow{\sim} V_2$ vector space isom. s.t. $\rho_2(g) \circ T = T \circ \rho_1(g), \forall g \in G$.

Lemma: Let ρ_V, ρ_W be rep of G on V and W , σ_V, σ_W corresponding reps of \mathfrak{g} .

Then (a) $V \oplus W$ is a rep $\rho = \rho_V \oplus \rho_W, \sigma = \sigma_V \oplus \sigma_W$

(b) V^* is a rep $\rho_V^*(g) = \rho_V(g^{-1})^* \quad \forall g \in G$

$$\sigma_V^*(x) = -\sigma_V(x)^* \quad \forall x \in \mathfrak{g}$$

(c) $V \otimes W$ is a rep, $\rho = \rho_V \otimes \rho_W$ $\rho(g) \left(\sum_i v_i \otimes w_i \right) = \sum_i \rho_V(g)v_i \otimes \rho_W(g)w_i$

$$\sigma(x) \left(\sum_i v_i \otimes w_i \right) = \sum_i \sigma_V(x)v_i \otimes w_i + v_i \otimes \sigma_W(x)w_i$$

(d) $\text{Hom}(V, W) = V^* \otimes W$ is a rep

$$\rho(g)(A) = \rho_W(g) \circ A \circ \rho_V(g^{-1}) \quad \forall g \in G, \forall A \in \text{Hom}(V, W)$$

Def: let ρ be a rep of G on $V \neq \{0\}$.

(a) V is irreducible if $W \subset V$ G -invt $\implies W = \{0\}$ or $W = V$.
subspace

(b) V is reducible if $\exists 0 \neq W \subsetneq V$ G -invt

(c) V is completely reducible if $V = \bigoplus_i V_i$ V_i irreducible.

Ex: $G = \mathbb{R}, V = \mathbb{R}^2, \rho_{\mathbb{R}}(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ is reducible but not completely reducible.