Lie groups. C3.5. HT26 [P. Bousseau]

Lecture 7 [09/02/26]

Def: G connected Lie group.

$\exists!$ (up to diffeo) simply connected manifold $\tilde{G}$

with covering map $\pi : \tilde{G} \to G$.

$\tilde{G}$: Universal cover of G.

$$\tilde{G} = \{ [\gamma] \text{ homotopy classes of path } \gamma : [0,1] \to G \text{ s.t. } \gamma(0) = e \}$$
with fixed $\gamma(1)$

$\pi : \tilde{G} \to G$

$[\gamma] \mapsto \gamma(1)$

$\tilde{G}$ is a Lie group: . identity $[\gamma_0(t) = e \quad \forall t]$

. Multiplication $[\gamma_1].[\gamma_2] = [\gamma_1(t)\gamma_2(t)]$

. Inverse $[\gamma]^{-1} = [\gamma(t)^{-1}]$

π : Lie group hom.

Ex: $G = S^1 = SO(2) = U(1) \Rightarrow \tilde{G} = \mathbb{R}$

Ex: $G = SO(3) \Rightarrow \tilde{G} = SU(2)$

$n \geq 3$

$\widetilde{SO(n)} =: Spin(n)$

Rem-

$$\begin{tikzcd} \tilde{G} \arrow[d, "F"] \arrow[dd, bend right, "\pi"'] \\ H \arrow[d, "f"] \\ G \end{tikzcd}$$

G, H connected f Lie group hom & covering

$\Rightarrow \exists!\ F : \tilde{G} \to H$ s.t. $f \circ F = \pi$

so $H = \tilde{G}/\mathrm{Ker}\, F$ In particular: $G = \tilde{G}/\underbrace{\mathrm{Ker}\,\pi}$

See end of last time. [normal discrete subgroup of $Z(\tilde{G})$]

Ex: $\widetilde{SL(2,\mathbb{R})}$ $\mathrm{Ker}\,\pi = \mathbb{Z}$

does not embed in any $GL(N,\mathbb{R})$ [but any <u>compact</u> Lie group does].

Thm: G, H Lie groups, G <u>simply connected.</u>

(a) $\exists$ 1-1 correspondence

$$\left\{ \begin{array}{c} \text{Lie algebra } \mathfrak{g} \to \mathfrak{h} \\ \text{hom} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Lie group hom} \\ G \to H \end{array} \right\}$$

$\xleftarrow{\quad}$ differential

(b) H simply connected, and $\mathfrak{g} \simeq \mathfrak{h} \Rightarrow G \simeq H$.

Proof: (a) $\Rightarrow$ (b) : Clear.

Proof of (a). $F : \mathfrak{g} \to \mathfrak{h}$ Lie algebra hom. Let $\mathfrak{k} = \{(X, F(X)) \mid X \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{h}$

Lie subalgebra (as F hom). $\qquad \mathfrak{g} \oplus \mathfrak{h} = \mathrm{Lie}(G \times H)$

Thm last time $\Rightarrow \exists!$ connected Lie group K of $G \times H$

s.t. $T_e K = \mathfrak{k}$

$$\begin{tikzcd} & K \subset G \times H \arrow[dl, "\pi = \pi_G|_K"'] \arrow[dr, "\pi_H|_K"] & \\ G & & H \end{tikzcd}$$

π Lie group hom & $d\pi_{(e,e)}$ isom $\Rightarrow$ π covering $\xRightarrow{G \text{ simply connected}}$ π diffeo

(↑ Thm last time)

Let $f = \pi_H|_K \circ \pi^{-1}$: Lie group hom with $df_e = F$.

G connected $\Rightarrow$ f unique
(generated by neighb of identity)

$\square$

Without proof:

Thm [Ado's Theorem] $\mathfrak{g}$ finite-dimensional Lie algebra ($/\mathbb{R}$)
$\Rightarrow \exists$ injective Lie alg hom $\mathfrak{g} \to \mathfrak{gl}(N, \mathbb{R}) = M_N(\mathbb{R})$ for some N

($\Rightarrow \exists$ connected Lie group $G \subset GL(N, \mathbb{R})$ s.t. $\mathfrak{g} = \mathrm{Lie}(G)$).

Thm [Lie 3rd Theorem] $\exists$ 1-1 correspondence:

$$\{\text{Lie algebras}\}/\text{isom} \leftrightarrow \left\{\begin{matrix}\text{Simply connected} \\ \text{Lie groups}\end{matrix}\right\}/\text{isom}.$$
(finite dim $/\mathbb{R}$)

Representation theory

Goal: Representation theory of compact Lie groups, very similar to representation theory of finite groups.

Def: A <u>representation</u> of a Lie group G on a (finite-dim) vertex space V is a Lie group hom $\rho: G \to \mathrm{Aut}(V)$ [<u>Note</u>: enough to check ρ continuous]

<u>Ex</u>: ρ representation of G on V. Action ϕ of G on functions on V:

$$\phi(g) f = f \circ \rho(g^{-1}) = f \circ \rho(g)^{-1}$$

<u>Ex</u>: U_n representation of $U(1)$ on $\mathbb{C}$, $\forall n \in \mathbb{Z}$:

$$\rho_n(\underset{U(1)}{\underset{\cap}{e^{i\theta}}}) \underset{\mathbb{C}}{\underset{\cap}{z}} = e^{in\theta} z$$

<u>Ex</u>: V_n $n \in \mathbb{N}$ representations of $SU(2)$

$$\overset{\shortparallel}{\mathbb{C}^{n+1}} = \mathrm{Span}\{z_1^n, z_1^{n-1} z_2, \dots, z_2^n\} \qquad z_1, z_2 \in \mathbb{C}$$

$$\rho(\underset{SU(2)}{\underset{\cap}{A}}) f = f \circ A^{-1} \qquad f = f(z_1, z_2)$$

<u>Rem:</u> ρ representation of G on $V \Rightarrow \sigma = d\rho_e : \mathfrak{g} \to \mathrm{End}(V)$ Lie algebra representation

<u>Ex</u>: U_n $G = U(1)$ $\mathfrak{g} = i\mathbb{R}$

$$\sigma(i\theta) z = \frac{d}{dt}\Big|_{t=0} \rho_n(e^{it\theta}) z = \frac{d}{dt}\Big|_{t=0} e^{int\theta} z = in\theta z$$

<u>Lemma:</u> G simply connected. $\forall \sigma: \mathfrak{g} \to \mathrm{End}(V)$ Lie algebra rep, $\exists \rho: G \to \mathrm{Aut}(V)$ rep of G on V s.t. $d\rho_e = \sigma$.

<u>Proof:</u> Previous Thm Today. $\square$

Ex: V_n representation of $SO(3) \Leftrightarrow n$ even.

Since $SO(3) = SU(2)/\{\pm I\}$ and $\rho(-I) = \rho(I) \Leftrightarrow n$ even.

Def: ρ_1, ρ_2: rep of G on V_1, V_2 are isomorphic if $\exists\, T: V_1 \xrightarrow{\sim} V_2$ vector space isom s.t. $\rho_2(g) \circ T = T \circ \rho_1(g), \ \forall g \in G$.

Lemma: Let ρ_V, ρ_W be rep of G on V and W, σ_V, σ_W corresponding reps of $\mathfrak{g}$.

Then (a) $V \oplus W$ is a rep $\rho = \rho_V \oplus \rho_W$, $\sigma = \sigma_V \oplus \sigma_W$

(b) V^* is a rep $\rho_V^*(g) = \rho_V(g^{-1})^* \quad \forall g \in G$

$$\sigma_V^*(x) = -\sigma_V(x)^* \quad \forall x \in \mathfrak{g}$$

(c) $V \otimes W$ is a rep, $\rho = \rho_V \otimes \rho_W$

$$\rho(g)\Big(\sum_i v_i \otimes w_i\Big) = \sum_i \rho_V(g) v_i \otimes \rho_W(g) w_i$$

$$\sigma(X)\Big(\sum_i v_i \otimes w_i\Big) = \sum_i \sigma_V(X) v_i \otimes w_i + v_i \otimes \sigma_W(X) w_i$$

(d) $\mathrm{Hom}(V, W) = V^* \otimes W$ is a rep

$$\rho(g)(A) = \rho_W(g) \circ A \circ \rho_V(g^{-1}) \quad \forall g \in G, \forall A \in \mathrm{Hom}(V, W)$$

Def: Let ρ be a rep of G on $V \neq \{0\}$.

(a) V is <u>irreducible</u> if $W \subset V$ G-invt subspace $\Rightarrow W = \{0\}$ or $W = V$.

(b) V is <u>reducible</u> if $\exists\, 0 \neq W \subsetneq V$ G-invt

(c) V is <u>completely reducible</u> if $V = \bigoplus_i V_i$ $\quad V_i$ irreducible.

Ex: $G = \mathbb{R}$, $V = \mathbb{R}^2$, $\rho(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$, $t \in \mathbb{R}$, is reducible but <u>not</u> completely reducible.