Lie groups. C3.5. HT26 [P. Bousseau]

Lecture 8 [13/02/2026]

$\rho: G \to \mathrm{Aut}(V)$

2 cases: • $V/\mathbb{R}$

• $V/\mathbb{C}$

Representations = rep   Irreducible representations = irrep

Note: Irrep $\Rightarrow$ Completely reducible.

Ex: All 1-dim reps are irreducible.   eg: rep U_n of $U(1)$ are irreps.

Ex: V_n irrep of $SU(2)$, but $V_n \otimes V_m$ not in general.

Ex: $U(1) \subset SU(2) \to V_n$ can be considered as a rep of $U(1)$

$\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$ V_n completely reducible$/U(1)$: $V_n = U_n \oplus U_{n-2} \oplus \cdots \oplus U_{-n}$

Lem (Schur's lemma) V, W irreps of Lie group G, $A: V \to W$ G-invt hom.

Then: a) $A = 0$ or A isomorphism

b) If $V, W/\mathbb{C} \Rightarrow A = \lambda I$ for some $\lambda \in \mathbb{C}$.

Proof: a) $\mathrm{Ker}\, A \subset V$ G-invt subspace $\Rightarrow \mathrm{Ker}\, A = \{0\}$ or V.

$A \neq 0 \Rightarrow A$ injective. $\mathrm{Im}\, A \subset W$ G-invt subspace

$\Rightarrow \mathrm{Im}\, A = 0$ ✗ or $\mathrm{Im}\, A = W$

$\Rightarrow A$ surjective.

b) A has an eigenvalue $\lambda \in \mathbb{C}$

$\mathrm{Ker}(A - \lambda I) \subset G$ G-invt V irrep $\Rightarrow \mathrm{Ker}(A - \lambda I) = \{0\}$ or V.

✗ ✓

□

Def: V $\mathbb{C}$-structure. $T: V \to V$ <u>real structure</u> on V if

$$T(\lambda_1 v_1 + \lambda_2 v_2) = \bar{\lambda}_1 v_1 + \bar{\lambda}_2 v_2, \quad \forall \lambda_1, \lambda_2 \in \mathbb{C},\ v_1, v_2 \in V,$$

and $T^2 = I$.

Let $U = \{u \in V \mid Tu = u\}$ real subspace of $V \to V = U \otimes_{\mathbb{R}} \mathbb{C}$

Then $T \leftrightarrow \mathbb{C}$-conjugation.

Ex: $T: \mathbb{C}^2 \to \mathbb{C}^2 \quad T(z_1, z_2) = (\bar{z}_2, -\bar{z}_1)$

T commutes with rep of $SU(2)$ on V_{2m}

Lem: If G is abelian, V complex rep, then $\dim V = 1$.

Proof: $g \in G \quad \rho: G \to \mathrm{Aut}(V)$ rep. Define $A: V \to V$, $v \mapsto \rho(g)v$

$$G \text{ abelian} \Rightarrow G\text{-invt} \underset{\uparrow}{\Rightarrow} A = \lambda_g I \quad \lambda_g \in \mathbb{C}$$

/$\mathbb{C}$ Schur lemma

$v \in V \setminus \{0\}, \quad \rho(g)v = \lambda_g v$

$$\Rightarrow \mathrm{Span}(v)\ G\text{-invt} \underset{\uparrow}{\Rightarrow} \mathrm{Span}(v) = V.$$

Schur Lemma

$\square$

Cor: G abelian, connected, compact. V real completely reducible rep

$$\Rightarrow V = \underbrace{\mathbb{R}^k}_{\text{Trivial}} \oplus V_1 \oplus \cdots \oplus V_m$$

$V_i \simeq \mathbb{R}^2$ rep of $SO(2)$

Ex: $V = \mathbb{R}^3$

$$G = SO(2) \simeq \left\{ \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \,\middle|\, A \in SO(2) \right\} \subset SO(3) \qquad \mathbb{R}^3 = \mathbb{R} \oplus \mathbb{R}^2 \circlearrowleft SO(2)$$

[Thm: G compact Lie group. V rep of G $\Rightarrow$ V completely reducible.

Def: V rep of G, $\rho: G \to \mathrm{Aut}(V)$. An inner product $\langle -,- \rangle$ is G-invariant if $\langle \rho(g)u, \rho(g)v \rangle = \langle u,v \rangle$ $\forall u,v \in V$, $\forall g \in G$.

V real: orthogonal rep
V complex: unitary

Prop: V rep of G. If V admits a G-invt inner product, then V is completely reducible.

Pf: Induction on $\dim V$. $\dim V = 1$: V irrep ✓

$\dim V$ If V irrep, ok. Else, $\exists\, W \subsetneq V$ G-invt, $W \neq 0$

$$\Rightarrow W^{\perp}\ G\text{-invt}\quad \Big[\langle \rho(g)v, w \rangle = \langle \rho(g)v, \rho(g)\rho(g^{-1})w \rangle \overset{\langle -,- \rangle\ G\text{-invt}}{=} \langle v, \underbrace{\rho(g^{-1})w}_{\in W} \rangle = 0 \quad \forall v \in W^{\perp}\ w \in W \Big] \ \square$$

So remains to show $\exists$ G-invt inner product.

Rem: For finite groups, $\langle -,- \rangle$ any inner product on V

$$\langle\langle u,v \rangle\rangle := \frac{1}{|G|} \sum_{g \in G} \langle \rho(g)u, \rho(g)v \rangle \qquad G\text{-invt}$$

Thm: G compact Lie group $\Rightarrow \exists!$ map $f \in C^0(G) \mapsto \int_G f \in \mathbb{R}$
with the following properties:

- $\int_G 1 = 1$
- $f > 0 \Rightarrow \int_G f > 0$
- $\int_G (\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \int_G f_1 + \lambda_2 \int_G f_2$ $\forall \lambda_1, \lambda_2 \in \mathbb{R}$, $f_1, f_2 \in C^0(G)$
- $\int_G f \circ L_h = \int_{g \in G} f(hg) = \int_G f \quad \forall h \in G$

Map is called Haar measure, unique normalized left-invariant integral.

Cor: $f \in C^0(G)$, G compact. $f \geq 0 \Rightarrow \int_G f \geq 0$ and $= 0 \Leftrightarrow f = 0$.

- $\int_G f \circ R_h = \int_G f \quad \forall h \in G$
- $\int_G f \circ i = \int_{g \in G} f(g^{-1}) = \int_G f$

$F: G \to H$ Lie group isom $f \in C^0(H)$: $\int_G f \circ H = \int_H f$.

Construction of a G-invt inner product. Pick $\langle -, - \rangle$ any inner product on G.

Define
$$\langle\langle u, v \rangle\rangle := \int_{g \in G} \langle \rho(g) u, \rho(g) v \rangle \quad \forall u, v \in V$$