B1.2 Set Theory
Sheet 3 — HT26

On this sheet, assume only ZF1-7, i.e. Extensionality, Empty Set, Pairing, Union, Power Set, Comprehension, and Infinity.

Section A

1. Let $\kappa, \lambda, \kappa', \lambda'$ be cardinalities with $\kappa \leq \kappa'$ and $\lambda \leq \lambda'$.
   (a) Show:
      (i) $\kappa + \lambda \leq \kappa' + \lambda'$.
      (ii) $\kappa \cdot \lambda \leq \kappa' \cdot \lambda'$.
   (b) Suppose moreover $\kappa < \kappa'$. Can we deduce strict inequalities in (i) and (ii)?

2. Show:
   (a) $|\mathbb{Q}| = \aleph_0$.
   (b) $|\mathbb{R}| = 2^{\aleph_0}$.
   (c) $|\mathbb{C}| = 2^{\aleph_0}$.

3. Let X be a non-empty countable set. Show that there exists a surjection $\mathbb{N} \to X$.

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Section B

4. Let κ, λ, and μ be cardinalities. Show that we have:

   (a) $(\kappa + \lambda) + \mu = \kappa + (\lambda + \mu)$

   (b) $(\kappa \cdot \lambda) \cdot \mu = \kappa \cdot (\lambda \cdot \mu)$

   (c) $\kappa \cdot (\lambda + \mu) = \kappa \cdot \lambda + \kappa \cdot \mu$

   (d) $\kappa^{\lambda+\mu} = \kappa^{\lambda} \cdot \kappa^{\mu}$

   (e) $\kappa^{\lambda \cdot \mu} = \left(\kappa^{\lambda}\right)^{\mu}$

   (f) $\left(\kappa \cdot \lambda\right)^{\mu} = \kappa^{\mu} \cdot \lambda^{\mu}$

5. (a) Let A, X, Y be sets such that $|X| \leq |A|$. Prove that $|X^Y| \leq |A^Y|$. Deduce that, for cardinalities κ, λ, μ, if $\kappa \leq \lambda$ then $\kappa^{\mu} \leq \lambda^{\mu}$.

   (b) Now let A, B, X, Y be sets with $|X| \leq |A|$ and $|Y| \leq |B|$. Prove that, apart from exceptional case(s), $|X^Y| \leq |A^B|$. What are the exceptional cases?

6. Calculate the cardinalities of the following sets, simplifying your answers as far as possible: your answer in each case should be a cardinality from the list $\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots$.

   In each case, first indicate why the set described exists.

   [*You may take for granted the existence and standard properties of the field $\mathbb{R}$ of real numbers, and may freely use the results in Question 1 and 4.*]

   (a) The set of all finite subsets of $\mathbb{N}$.

   [*Hint: Consider binary expansions of natural numbers (properties of which you may use without proof).*]

   (b) The set of all finite sequences of natural numbers, i.e. $\bigcup\{\mathbb{N}^n : n \in \mathbb{N}\}$.

   [*Do not try to prove that a countable union of countable sets is countable – this is not a consequence of ZF. Use (a), or alternatively use (without proof) standard facts from arithmetic.*]

   (c) The set of functions $f : \mathbb{R} \to \mathbb{R}$.

   (d) The set of continuous functions $f : \mathbb{R} \to \mathbb{R}$.

   [*You may use without proof that a continuous function is determined by its values on $\mathbb{Q}$.*]

   (e) The set of equivalence relations on $\mathbb{N}$.

7. Let $f : X \to Y$ be surjective. Prove that $|\mathcal{P}(Y)| \leq |\mathcal{P}(X)|$.

   [*You should not assume that there exists an injective map $g : Y \to X$.*]

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8. In this question, $+$ and $\cdot$ denote cardinal addition and multiplication (but n^+ still denotes the successor $n \cup \{n\}$ of n).
   (a) Let $n \in \mathbb{N}$. Prove that $n^+ = n + 1$.
   (b) Let κ be any cardinality and $n \in \mathbb{N}$. Prove:
      (i) $\kappa + 0 = \kappa$
      (ii) $\kappa \cdot 0 = 0$
      (iii) $\kappa \cdot n^+ = \kappa \cdot n + \kappa$
   (c) We now have two definitions of $n + m$ for $n, m \in \mathbb{N}$: the definition by recursion on m, and the definition in terms of cardinalities. The same goes for multiplication. Prove in each case that the two definitions agree.

9. (a) Let X be a well-ordered set, and let $x \in X$. Show that either x is the greatest element in X or x has an immediate successor (that is, an element $x^* \in X$ with $x < x^*$ such that there is no $y \in X$ with $x < y < x^*$).
   (b) Let $X \subseteq \mathbb{R}$ be such that the inherited order $<$ from $\mathbb{R}$ is a well-order on X. Prove that X must be countable. [Hint: consider the intervals (x, x^*).]

10. Say a set X is *well-orderable* if there exists a well-order $\preceq$ on X.
   (a) Show that any countable set is well-orderable.
   (b) Let $(X, <)$ be a totally ordered set, and suppose X is well-orderable. Show that the following are equivalent:
      (i) $(X, <)$ is well-ordered;
      (ii) X has no infinite descending sequence, i.e. there is no function $f : \mathbb{N} \to X$ such that $f(n^+) < f(n)$ for all $n \in \mathbb{N}$.

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Section C

11. Fill in the gaps in the "alternative" proof of Cantor-Schröder Bernstein sketched in lectures: let $f : X \to Y$ and $g : Y \to X$ be injections, and find a subset $A \subseteq X$ with $g[Y \setminus f[A]] = X \setminus A$ using recursion on $\mathbb{N}$ rather than Tarski's Fixed Point Theorem.

12. A complex number $a \in \mathbb{C}$ is **algebraic** if $f(a) = 0$ for some non-zero integer polynomial $f \in \mathbb{Z}[x] \setminus \{0\}$, otherwise it is **transcendental**. Prove that the algebraic complex numbers form a countable set, and conclude that transcendental real numbers exist.

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