Lie groups. C3.5. HT26. [P. Bousseau]

Lecture 9 [16/02/2026]

ρ rep of Lie group G on (finite-dim) vector space / $\mathbb{R}$ or $\mathbb{C}$.
$\mathbb{F}$ //

Today: Characters.

Def: The character χ_V of V is: $\chi_V : G \to \mathbb{F}$
$g \mapsto \chi_V(g) := \mathrm{tr}(\rho(g))$

Ex: U_n rep of $U(1)$ $n \in \mathbb{Z}$
$t = e^{i\theta} \in U(1)$ $\chi_{U_n}(t) = t^n$

Prop:
- $\chi_V : G \to \mathbb{F}$ smooth function
- $\chi_V(hgh^{-1}) = \chi_V(g)$
- $\chi_{V \oplus W} = \chi_V + \chi_W$
- $\chi_{V^*}(g) = \chi_V(g^{-1})$
- $\chi_V(e) = \mathrm{Tr}_V \, \mathrm{id} = \dim V$
- $V \simeq W \Rightarrow \chi_V = \chi_W$
- $\chi_{V \otimes W} = \chi_V \chi_W$
- If V unitary or orthogonal: $\chi_{V^*}(g) = \overline{\chi_V(g)}$.
[Always if G compact by last time]

Proof: See Pb Sheet. $\square$

Ex: V_n rep of $U(1) = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\} < SU(2)$

$$V_n = U_n \oplus U_{n-2} \oplus \cdots \oplus U_{2-n} \oplus U_{-n}$$
$$\chi_{V_n}(t) = t^n + t^{n-2} + \cdots + t^{2-n} + t^{-n}$$

Def: $V^G = \{v \in V \mid \rho(g)v = v \quad \forall g \in G\}$

Prop: $V^G = \left\{ \int_{g\in G} \rho(g) w \mid w \in V \right\}$ for G <u>compact</u>.

Proof: let $W = \left\{ \int_{g \in G} \rho(g) w \mid w \in V \right\}$ subspace of W

$h \in G$

$$\rho(h)\int_{g\in G}\rho(g)w \underset{\rho(h)\text{ linear}}{=} \int_{g\in G}\rho(h)\rho(g)w \underset{\rho\text{ hom}}{=} \int_{g\in G}\rho(hg)w \underset{\text{left-invt of }\int}{=} \int_{g\in G}\rho(g)w$$

so $W \subset V^G$

Conversely, if $v \in V^G$, $\int_{g\in G}\rho(g)v = \int_{g\in G} v = \left(\int_{g\in G} 1\right) v = v$. $\square$

Lem: G compact $\dim V^G = \int_{g\in G} \chi_V(g)$

Proof: Define $p: V \to V$ projection onto V^G

$$v \mapsto \int_{g\in G} \rho(g)v$$

$$\underset{\dim V^G}{\overset{}{\operatorname{tr}(p)}} \underset{\text{tr linear}}{=} \int_{g\in G} \operatorname{tr}\rho(g) = \int_{g\in G}\chi_V(g)$$

$\operatorname{tr}(p) = \dim V^G$

$\square$

Def: V, W reps of G. $\operatorname{Hom}_G(V,W) := \{G\text{-invt hom } V \to W\}$
$= (\operatorname{Hom}(V,W))^G$

Thm: G compact, V, W rep

$$\langle \chi_V, \chi_W\rangle := \int_G \overline{\chi}_V \chi_W = \dim \operatorname{Hom}_G(V,W)$$

Proof: $\mathrm{Hom}(V,W) = V^* \otimes W$

Previous lemma $\Rightarrow \dim \mathrm{Hom}_G(V,W) = \dim (V^* \otimes W)^G$

$$= \int_G \chi_{V^* \otimes W} = \int_G \chi_{V^*} \chi_W = \int_G \overline{\chi_V}\, \chi_W$$
$$= \langle \chi_V, \chi_W \rangle$$
$\square$

G compact:

Cor: • If $V \not\cong W$: $\langle \chi_V, \chi_W \rangle = 0$
V, W irreducible

• $\chi_V = \chi_W \Rightarrow V \simeq W$

• $\langle \chi_V, \chi_V \rangle = 1$ if $\mathbb{F} = \mathbb{C}$, $\langle \chi_V, \chi_V \rangle \geq 1$ if $\mathbb{F} = \mathbb{R}$

Proof: Schur's Lemma. $\square$

Thm: G compact V, W rep $\chi_V = \chi_W \Rightarrow V \simeq W$.

Proof: V completely irreducible $V = \bigoplus_i V_i^{n_i}$ V_i irreducible distinct

$$\chi_V = \sum_i n_i \underbrace{\chi_{V_i}}_{\perp} \qquad W = \bigoplus_i V_i^{m_i}$$

$$= \sum_i m_i \chi_{V_i} \qquad n_i = \langle \chi_V, \chi_{V_i} \rangle = \langle \chi_W, \chi_{V_i} \rangle = m_i$$

Rem: Familiar from finite groups (which are examples of compact Lie groups). $\square$

Thm: G compact. V, W irreducible rep with G-invt inner products $\langle \cdot,\cdot \rangle_V, \langle \cdot,\cdot \rangle_W$

Define for $v_1, v_2 \in V$, $w_1, w_2 \in W$

$$F(v_1, v_2, w_1, w_2) = \int_{g \in G} \langle \rho(g) v_1, v_2 \rangle_V \overline{\langle \rho(g) w_1, w_2 \rangle_W}$$

Then a) $V \not\simeq W \Rightarrow F = 0$.

b) $V \simeq W$, $/\mathbb{C}$ $\Rightarrow F(v_1, v_2, w_1, w_2) = \dfrac{\langle v_1, w_1\rangle_V \langle v_2, w_2\rangle_V}{\dim V}$

<u>Def</u>: V unitary/orthogonal rep of G

$\{v_1, \dots, v_n\}$ unitary/orthonormal basis of V

→ define <u>matrix coefficients</u>: $\rho_{ij} : G \longrightarrow \mathbb{F}$

$$g \longmapsto \langle \rho(g) v_i, v_j \rangle$$

<u>Thm</u> ⇒ V irred, G compact ⇒ the ρ_{ij} are L^2-orthogonal functions on G.

<u>Proof of Thm</u>: Skip (use Schur's Lemma). □

<u>Ex</u>: $G = U(1) = S^1$ matrix coeff: $e^{in\theta}$ $n \in \mathbb{Z}$

⇒ Matrix coeff for a complete orthogonal basis for $L^2(S^1)$ (Fourier series).

<u>Thm</u> [Peter-Weyl] G compact Lie group [Without proof]

$$L^2(G) \simeq \widehat{\bigoplus_{V}} V \otimes V^*$$

$G \times G$ left-right ↻ $L^2(G)$; V irreducible rep of G; Left (V); Right (V^*)

<u>Rem</u>:

$\chi_V \longleftrightarrow \mathrm{id}_V \in V \otimes V^*$

$f_{ij} \in L^2(G) \longleftrightarrow v_i \otimes v_j^*$