Lie groups. C3.5 HT26. [P. Bousseau]

Lecture 10 [20/02/2026]

Today: Maximal Tori G: compact connected Lie group.

Ex: $T^n = (S^1)^n$ torus, abelian.

Def: T is a torus in G if T is a subgroup of G with $T \simeq T^n$ for some n.
T is a maximal torus if $\forall\, T \subset T'$ tori in G, $T = T'$.

Rem: T compact $\Rightarrow$ closed in $G \Rightarrow$ Automatically Lie subgroup. [embedded]

Ex: $G = U(n)$ $T = \{ \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \mid \theta_j \in \mathbb{R} \}$ is a max torus of $U(n)$
$\simeq T^n$

$G = SU(n)$ $T = \{ \operatorname{diag}(e^{i\theta_1}, \dots, e^{i\theta_n}) \mid \theta_1 + \dots + \theta_n = 0 \}$ — $SU(n)$
$\simeq T^{n-1}$

Lem: T torus in G

(a) T embedded Lie group

(b) $\exists\, T'$, $T \subset T'$ max torus

(c) T max torus $\Rightarrow gTg^{-1}$ max torus $\forall g \in G$.

(d) T max torus $\Leftrightarrow$ T max connected abelian subgroup of G.

Pf: a) T compact $\Rightarrow T$ closed in $G \Rightarrow T$ embedded Lie subgroup.

b) Torus $\to$ Lie subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$

$T \subset T'$ torus $\Rightarrow \dim T' > \dim T$ (think of Lie algebras)

$\Rightarrow$ Process of increasing abelian Lie subalgebras terminates as $\dim \mathfrak{g} < \infty$.

c) $gTg^{-1} \subset T' \overset{b)}{\Rightarrow} T \subset g^{-1} T' g$ torus

T_{max} $\qquad$ T max $\Rightarrow T = g^{-1} T' g \Rightarrow T' = gTg^{-1}$.

d) See PS sheet. $\square$

Ex: $G = SO(2n)$ $\quad T = \left\{ \begin{pmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_n \end{pmatrix} \middle| R_j \in SO(2) \right\} \simeq T^n$ max torus

$G = SO(2n+1)$

$$T = \left\{ \begin{pmatrix} R_1 & & & \\ & \ddots & & \\ & & R_n & \\ & & & 1 \end{pmatrix} \middle| R_j \in SO(2) \right\} \simeq T^n \text{ max torus}$$

Def: Let $T \subset G$ max torus.

The Weyl group W of T is $W = N(T)/T$

$$= \{ g \in G \mid gTg^{-1} \} / \underset{\text{normal subgroup}}{T}$$

Rem: Will show: all max tori of G are conjugate $\Rightarrow$ W indep of T up to isom: "Weyl group of G"

Ex: $G = SU(2)$ $\quad T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$

$$W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \simeq \mathbb{Z}/2\mathbb{Z}$$

Prop: W is finite.

Pf: $N_0 :=$ connected component of Id in $N(T)$

$\Rightarrow N(T)/N_0$ | discrete + compact $\Rightarrow N(T)/N_0$ finite.

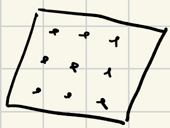
previously

Goal: $N_0 = T$.

(see Pb Sheet)

the action of $N(T)$ on T & $\mathfrak{t} = \mathrm{Lie}(T)$ commutes with exp

$\Rightarrow N(T)(\mathrm{Ker}\exp) \subset \mathrm{Ker}\exp$ discrete (see before)



N_0 connected $\rightarrow$ $\Rightarrow N_0 |_{\mathrm{Ker}\exp} = Id$

$\Rightarrow N_0$ acts trivially on $\mathfrak{t}$

$\Rightarrow N_0$ acts trivially on T

$$g t g^{-1} = t \qquad gt = tg \quad \forall g \in N_0$$

Pick a 1-parameter subgroup of N_0

$\Rightarrow T \subset \langle T, \dots \rangle$

T max abelian $\Rightarrow T = N_0$.

$\square$

Thm: G compact connected Lie group.

a) T max torus $\Rightarrow G = \{gtg^{-1}, g \in G, t \in T\}$

b) If T' is a max torus, $\exists g \in G$ s.t. $T' = gTg^{-1}$.

Def: T torus in G. Then $t \in T$ is a <u>generator</u> of T if

$$\overline{\{t^n \mid n \in \mathbb{Z}\}} = T$$

Generators always exist (think about irrational slopes).

· Proof of a) $\Rightarrow$ b): T' max torus. Let $t' \in T'$ generator.

$$\exists g \in G, t \in T \text{ s.t. } t' = gtg^{-1} \in gTg^{-1}$$

$$\text{then } (t')^n = gt^ng^{-1} \in gTg^{-1}$$

$$\text{Closure} \Rightarrow T' \subset gTg^{-1} \quad \text{Closed}$$

$$T' \text{ max} \Rightarrow T' = gTg^{-1}. \qquad \square$$

Proof of a)? See next time.