Lie groups C3.5. HT 26 [P. Bousseau]

Lecture 11 [23/02/2026]

Proof of Thm a) (difficult, only sketch below)

Consider $\tilde{F} : G \times T \to G$ smooth
$(g, s) \mapsto g s g^{-1}$

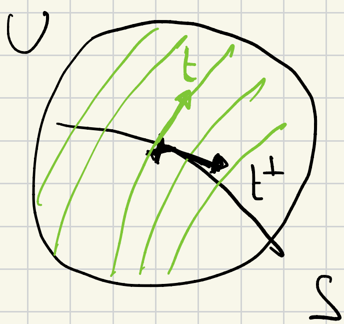
Step 1: G/T manifold (⚠ not a group, T not normal) and $F : G/T \times T \to G$ well-defined

Choose a G-invt inner product on $\mathfrak{g}$: $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{t}^\perp$ (G compact: Ad is orthogonal)

Implicit function thm: $\exists V \ni 0$ open in $\mathfrak{t}^\perp$, $U \ni e$ open in G

s.t. $\exp(V) = S$

s.t. $S \cap \{gt \mid t \in T\}$ = single pt or ϕ $\forall g \in G$



S : chart for G/T near e

Left mult $\Rightarrow$ G/T manifold

$\dim G/T = \dim \mathfrak{t}^\perp = \dim G - \dim T$.

Step 2:

$\{t_1, \dots, t_k\}$ oriented orthonormal basis of $\mathfrak{t}$
extend to $\{t_1, \dots, t_k, v_1, \dots, v_n\}$ — $\mathfrak{g}$
(T & G oriented)
$\Rightarrow$ Orientation on $\mathfrak{t}^\perp$

$\Rightarrow$ Orientation on G/T

Inner product on $\mathfrak{t}^\perp$ T-invt
& T connected $\Rightarrow$

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Step 3. $t \in T$ generator $\Rightarrow |F^{-1}(t)| = |W|$

$$(gT, s) \in F^{-1}(t) \iff gsg^{-1} = t$$

$$gs^n g^{-1} = t^n$$

$$\Rightarrow gTg^{-1} = T$$

t gen + T max

$$\Rightarrow g \in N(T)$$

$$N(T)/T \simeq W.$$

Step 4: $t \in T$ generator $\Rightarrow t$ regular value of F

$$dF_{(gT,s)}(X, Y) = Ad(g)\left(Ad(s^{-1})X - X + Y\right)$$

$(gT, s) \in F^{-1}(t)$ $\qquad X \in \mathfrak{t}^{\perp} \simeq T_{gT}(G/T)$

$$Y \in \mathfrak{t} \simeq T_s T$$

$Ad(g) \in O(n)$ $\quad$ G connected $\Rightarrow Ad(g) \in SO(n)$

so: $\quad \det dF_{(gT,s)} = \det_{\mathfrak{t}^{\perp}}\left(Ad(s^{-1}) - I\right)$

T max abelian $\Rightarrow 1$ not eigenvalue of $Ad(s^{-1})$
(else construct a bigger group)

$\Rightarrow \det dF_{(gT,s)} \neq 0 \Rightarrow t$ regular value
& sign $\det dF_{(gT,s)}$ independent of $(gT,s) \in F^{-1}(t)$

Step 5: Smooth map between compact oriented manifolds

$$\deg(F) = \sum_{(gT,s) \in F^{-1}(t)} \operatorname{sgn} \det dF_{(gT,s)} \neq 0 \;\Big|\; \Rightarrow F \text{ surjective.} \quad \square$$

<u>Cor:</u> $\exp: \mathfrak{g} \to G$ surjective. Pf: T connected abelian
$\Rightarrow \exp: \mathfrak{t} \to T$ surjective

$$G = \{ g \exp(X) g^{-1} \mid g \in G, X \in \mathfrak{t}$$
$$= \{ \exp(\mathrm{Ad}(g) X) \mid g \in G, X \in \mathfrak{t} \} \quad \square$$

<u>Def:</u> <u>Rank</u> of $G :=$ dim T $\quad T \subset G$ max torus.

<u>Ex:</u> $\mathrm{rk}\, U(n) = n \qquad \dim U(n)/T = n^2 - n = n(n-1)$ even!

<u>Prop:</u> $\dim G/T$ is even.

<u>Proof:</u> $\mathrm{Ad}|_T$ is a representation of T on $\mathfrak{g}$ $\to$ Hence on $\mathfrak{t}^\perp$

T connected compact abelian $\Rightarrow \mathfrak{t}^\perp \simeq \mathbb{R}^m \oplus \bigoplus_i V_i \leftarrow$ 2-dim irrep

$\mathrm{Ad}(t)$ cannot have eigenvalues ± 1 on $\mathfrak{t}^\perp \Rightarrow m = 0$.
$t \in T$ generator
$\in O(\mathfrak{t}^\perp)$ $\quad \square$

<u>Def:</u> $\mathrm{Ad}|_T$ on $\mathfrak{g}$ decomposes as: $\mathfrak{g} \simeq \mathfrak{t} \oplus \bigoplus_\alpha \mathfrak{g}_\alpha$
$\qquad \uparrow$ 2-dim irrep of T

If $(e^{2i\pi x_1}, \dots, e^{2i\pi x_k}) \in T$, then action on $\mathfrak{g}_\alpha \simeq \mathbb{C}$
is $e^{2i\pi\theta_\alpha}$ where $\theta_\alpha = m_1 x_1 + \dots + m_k x_k$,
$m_1, \dots, m_k \in \mathbb{Z}$

<u>Roots</u> of G are $\pm\theta_\alpha : \mathfrak{t} \to \mathbb{R}$, $\pm\theta_\alpha \in \mathfrak{t}^*$.

Ex:

G	T	Roots
$U(n)$	$\begin{pmatrix} e^{2i\pi x_1} & & \\ & \ddots & \\ & & e^{2i\pi x_n}\end{pmatrix}$	$\pm(x_j - x_k) \quad j<k$
$SU(n)$	— $\sum_j x_j = 0$	$\pm(x_j - x_k) \quad j<k$
$SO(2m)$	$\begin{pmatrix} R_1 & & 0 \\ & \ddots & \\ 0 & & R_m\end{pmatrix}$ $R_j = \begin{pmatrix} \cos 2\pi x_j & -\sin 2\pi x_j \\ \sin 2\pi x_j & \cos 2\pi x_j \end{pmatrix}$	$\pm x_j \pm x_k$ $j<k$
$SO(2m+1)$	$\begin{pmatrix} R_1 & & & 0 \\ & \ddots & & \\ & & R_m & \\ 0 & & & 1\end{pmatrix}$	$\pm x_j \pm x_k$ $j<k$ and $\pm x_j$

Lemma: W permutes the roots.

Pf: $g \in N_G(T)$, $t \in T$, define $\rho(t) = Ad(gtg^{-1})$ rep of T on $\mathfrak{g}$

$$\overset{\shortparallel}{Ad(g)\,Ad(t)\,Ad(g)^{-1}}$$

so ρ is isomorphic to $Ad|_T$

$\Rightarrow$ Same decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_a \mathfrak{g}_a$ hence permutes the roots. $\square$

Ex: $SU(2)$ 2 roots $W = \mathbb{Z}/2\mathbb{Z}$ Claim: W swap the 2 roots.

$N_G(T) \curvearrowright \mathfrak{g}$

$W \curvearrowright \mathfrak{t}$

(T acts trivially on $\mathfrak{t}$)

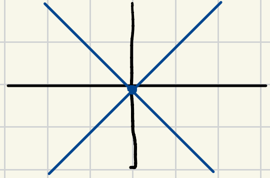
Def: Let $\pm\theta_a$ a root of G.

- $\operatorname{Ker}\theta_a \subset \mathfrak{t}$ <u>root hyperplane</u>.
- $\Theta_a : T \longrightarrow S^1$ <u>root homomorphism</u>

$$\overset{\shortparallel}{e^{2\pi\theta_a}}$$

Rem: $X \in \operatorname{Ker}\theta_a \Rightarrow e^{2\pi X} \in \operatorname{Ker}\Theta_a$

<u>Ex</u>: $SO(4)$ $rk = k = 2$ Roots: $\pm x_1 \pm x_2$



<u>Prop</u>: $W \hookrightarrow O(k)$ $k = \text{rank } G$

(no proof) $= O(t)$

<u>Thm</u>: Every reflection in a root hyperplane lies in W.

<u>Thm</u>: W is generated by reflections in root hyperplanes.

(No proof).