Lie groups C3.5 HT26 [P. Bousseau]

Lecture 12. [27/02/2026]

G compact connected Lie group $\rightarrow$ $\mathfrak{g}$: Lie algebra

$T \subset G$ max torus $\rightarrow$ $\mathfrak{t} \subset \mathfrak{g}$

$\boxed{\text{Rank of } G = \mathrm{rk}\, G := \dim T = r}$

Ad : G acts on $\mathfrak{g}$
$\cup$
$\mathrm{Ad}|_T$: T acts on $\mathfrak{g}$

Decompose $\mathfrak{g}$ as irreducible reps of T:

$$\mathfrak{g} = \underbrace{\mathfrak{t}}_{\substack{\text{Trivial rep}\\ \text{of } T}} \oplus \bigoplus_a \underbrace{\mathfrak{g}_a}_{= \mathbb{R}^2 \simeq \mathbb{C}}$$

2-dim irred rep of T

$t = \left(e^{2i\pi x_1}, \dots, e^{2i\pi x_r}\right)$ acts as $e^{2i\pi \theta_a(x)}$

$\theta_a(x) = m_1 x_1 + \dots + m_r x_r$
$m_1, \dots, m_r \in \mathbb{Z}$

$\theta_a : \mathfrak{t} \rightarrow \mathbb{R}$
$x \mapsto \theta_a(x)$ $\qquad \theta_a \in \mathfrak{t}^*$

<u>Roots</u>: $\{\pm \theta_a\} \subset \mathfrak{t}^*$

Alternative viewpoint:

$$\underbrace{\mathfrak{g} \otimes \mathbb{C}}_{\mathbb{C}\text{-rep}} = (\mathfrak{t} \otimes \mathbb{C}) \oplus \left(\bigoplus_a \mathfrak{g}_{\theta_a} \oplus \mathfrak{g}_{\theta_{-a}} \right)$$

$\mathfrak{g}_{\theta_a} = \mathbb{C}$, acted on by $e^{2i\pi\theta_a(x)}$; $\mathfrak{g}_{\theta_{-a}} = \mathbb{C}$, acted on by $e^{-2i\pi\theta_a(x)}$

Root hyperplanes: $\quad \ker \theta_a \subset \mathfrak{t}$

Weyl group $W = N_G(T)/T \underset{\text{Thm}}{=}$ group generated by orthogonal reflections / root hyperplanes $\ker \theta_a$

$W \circlearrowright \mathfrak{t}$

<u>Ex</u>: $G = U(2)$ $\qquad\qquad$ $\dim U(2) = 4$

$$T = \left\{ \begin{pmatrix} e^{2i\pi x_1} & 0 \\ 0 & e^{2i\pi x_2} \end{pmatrix} \middle| \; x_1, x_2 \in \mathbb{R} \right\} \qquad \mathrm{rk}\, U(2) = \dim T = 2$$

$$\mathfrak{g} = \left\{ \begin{pmatrix} ia & b \\ -\bar{b} & ic \end{pmatrix} \middle| \; a, c \in \mathbb{R},\ b \in \mathbb{C} \right\}$$

$$\| $$

$$\underbrace{\left\{ \begin{pmatrix} ia & 0 \\ 0 & ic \end{pmatrix} \right\}}_{= \mathfrak{t} \simeq \mathbb{R}^2} \oplus \underbrace{\left\{ \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \middle| \; b \in \mathbb{C} \right\}}_{\mathfrak{g}_a}$$

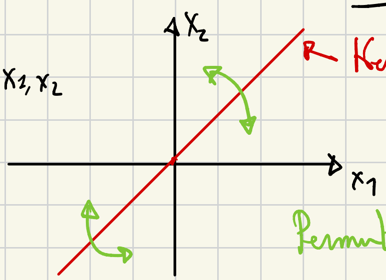
$$t = \begin{pmatrix} e^{2i\pi x_1} & 0 \\ 0 & e^{2i\pi x_2} \end{pmatrix}$$

$$t \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} t^{-1} = \begin{pmatrix} e^{2i\pi x_1} & 0 \\ 0 & e^{2i\pi x_2} \end{pmatrix} \begin{pmatrix} 0 & b \\ -\bar{b} & 0 \end{pmatrix} \begin{pmatrix} e^{-2i\pi x_1} & 0 \\ 0 & e^{-2i\pi x_2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e^{2i\pi(x_1 - x_2)} b \\ -e^{-2i\pi(x_1-x_2)} \bar{b} & 0 \end{pmatrix}$$

$b \mapsto e^{2i\pi(x_1-x_2)} b$ so <u>roots</u> are $\pm(x_1 - x_2)$.

$\mathfrak{t} \simeq \mathbb{R}^2_{x_1, x_2}$



$\ker \theta_a$ = Root hyperplane $x_1 = x_2$

$W = \mathbb{Z}/2\mathbb{Z}$

Permutation of x_1 and x_2 = Reflection$/\{x_1 = x_2\}$

Ex: $G = SU(3)$ $\quad \dim = 8$

$$T = \left\{ \begin{pmatrix} e^{2i\pi x_1} & & \\ & e^{2i\pi x_2} & \\ & & e^{2i\pi x_3} \end{pmatrix} \,\middle|\, \begin{array}{l} x_i \in \mathbb{R} \\ x_1 + x_2 + x_3 = 0 \end{array} \right\}$$

$\operatorname{rk} G = \dim T = 2$

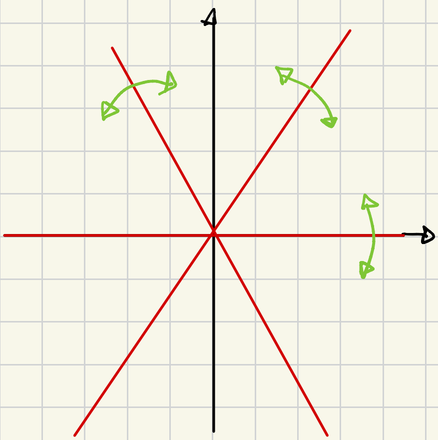
$\mathfrak{t} = \mathbb{R}^2$

$$\underbrace{\mathfrak{su}(3)}_{\dim 8} = \underbrace{\mathfrak{t}}_{\dim 2} \oplus \underbrace{\ ?\ }_{\dim 6}$$

6 roots.

$\Downarrow$

3 roots hyperplanes



$W = S_3$ permuting (x_1, x_2, x_3)

generated by 3 transpositions

$\Updownarrow$

3 reflections / 3 root hyperplanes.

The representation ring

G compact connected Lie group

T: max torus.

Restrict to $\mathbb{C}$-rep V.

Notation: $nV := \underbrace{V \oplus \cdots \oplus V}_{n \text{ times}}$

Def: The representation ring of G is $R(G) = \left\{ \sum_i n_i V_i \;\middle|\; \begin{array}{l} n_i \in \mathbb{Z} \\ \text{finitely many} \neq 0 \\ V_i \text{ non-isom} \\ \mathbb{C}\text{-irred reps} \\ \text{of } G \end{array} \right\}$

with. sum operation: $\oplus$

. Multiplication: $\otimes$

Character ring of G:

$$\chi(R(G)) = \left\{ \sum_i n_i \chi_i \;\middle|\; \begin{array}{l} n_i \in \mathbb{Z}, \text{only finitely many} \neq 0 \\ V_i \text{ non-isom } \mathbb{C}\text{-irred reps of } G \end{array} \right\}$$

$+$, $\times$.

$$R(G) \xrightarrow{\sim} \chi(R(G))$$

$$\sum_i n_i V_i \longmapsto \sum_i n_i \chi_{V_i} \quad \text{ring isom}$$

Ex: $G = U(1) \ni t = e^{i\theta}$ $\quad$ $\mathbb{C}$-irred rep: U_n $\quad$ $\chi_{U_n}(t) = t^n$

$$R(U(1)) \simeq \mathbb{Z}[t, t^{-1}]$$

Def: V $\mathbb{C}$-irred rep of $G \Rightarrow V = \bigoplus_\alpha V_\alpha$ as $\mathbb{C}$-irred reps of T

$\uparrow$ 1-dim

$$\chi_{V_\alpha}(e^{2i\pi x_1}, \dots, e^{2i\pi x_k}) = e^{2i\pi(m_1 x_1 + \cdots + m_k x_k)}$$

$m_j \in \mathbb{Z}$

$\sum_{j=1}^{k} m_j x_j$ are the <u>weights</u> of rep V. $\in \mathfrak{t}^* \circlearrowleft W$

$t_1^{m_1} \cdots t_k^{m_k}$

<u>Rem:</u> <u>Roots</u> = Weights of Ad on $\mathfrak{g} \otimes \mathbb{C}$.

<u>Lemma:</u> W permutes the weights.

<u>Proof:</u> Same as for roots. $\square$

<u>Ex:</u> $T \simeq T^k \qquad R(T) = \mathbb{Z}[t_1^{\pm}, \dots, t_k^{\pm}]$

<u>Prop:</u> V $\mathbb{C}$-irred rep of $G \Rightarrow \chi_V$ determined by $\chi_V|_T$ and $\chi_V|_T$ is invt / Weyl group.

So: map $\chi(R(G)) \to \chi(R(T))^W$ is injective

$$\chi_V \longmapsto \chi_V|_T$$

<u>Proof:</u> $h \in G$ $\exists t \in T, g \in G$ s.t. $h = gtg^{-1} \Rightarrow \chi_V(h) = \chi_V(t)$

$$= \chi_V|_T(t)$$

W permutes the weights $\Rightarrow \chi_V|_T$ W-invt. $\square$

<u>Ex:</u> $G = SU(2) \qquad T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$

$W = \mathbb{Z}/2\mathbb{Z} : t = e^{i\theta} \mapsto t^{-1} = e^{-i\theta}$

$\chi(R(T))^W = \left\{ \sum_j m_j (t^j + t^{-j}) \mid m_j \in \mathbb{Z} \right\} = \mathbb{Z}[t + t^{-1}]$

All irred rep of $SU(2)$ are V_n's!

$V = V_n \Rightarrow$ Surjective $\Rightarrow R(SU(2)) \simeq R(T)^W \simeq \mathbb{Z}[t + t^{-1}]$