B1.2 Set Theory

Sheet 4 — HT26

On this sheet, assume ZF.

Section A

1. Prove the following.

   (a) (i) $0 = \emptyset$ is an ordinal.

   (ii) If α is an ordinal, then so is $\alpha^+ = \alpha \cup \{\alpha\}$.

   (iii) If Γ is a set of ordinals, then $\bigcup \Gamma$ is an ordinal.

   (b) Every $\beta \in \mathbf{ON}$ is of precisely one of the following three types:

   - Zero ordinal: $\beta = 0$.
   - Successor ordinal: $\beta = \alpha^+$ for some $\alpha \in \mathbf{ON}$.
   - Limit ordinal: $\beta = \bigcup \beta$ and $\beta \neq 0$.

2. Show that AC is equivalent to every set X having a choice function, a function $h : \mathcal{P}(X) \setminus \{\emptyset\} \to X$ such that $h(A) \in A$ for all $\emptyset \neq A \subseteq X$.

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Section B

3. Consider the ordinals:

$$\alpha_1 = (\omega+1)\cdot 2, \quad \alpha_2 = 2\cdot(\omega+1), \quad \alpha_3 = (\omega+1)\cdot\omega, \quad \alpha_4 = \omega\cdot(\omega+1).$$

Determine the ordering between these ordinals, i.e. determine for which i, j we have $\alpha_i < \alpha_j$.

Complement your answer with an illustrative diagram of the largest of these ordinals, indicating where the smaller ones lie within it.

4. (a) Let $\alpha, \beta \in \mathbf{ON}$. Show that $(\alpha\cdot\beta, \in) \cong (\alpha, \in) \times (\beta, \in)$ (where recall the right hand side denotes the reverse lexicographic product order).

   [*You may use the corresponding result (proven in lectures) for ordinal addition and sum orders.*]

   (b) Let α, β, γ be ordinals. Show:

   (i) $(\alpha\cdot\beta)\cdot\gamma = \alpha\cdot(\beta\cdot\gamma)$.

   (ii) $\alpha\cdot(\beta+\gamma) = \alpha\cdot\beta + \alpha\cdot\gamma$.

   (iii) $\alpha \neq 0,\ \beta < \gamma \ \Rightarrow\ \alpha\cdot\beta < \alpha\cdot\gamma$.

   (iv) $\alpha \leq \gamma \ \Rightarrow\ \alpha\cdot\beta \leq \gamma\cdot\beta$.

   (v) $\alpha < \gamma \ \not\Rightarrow\ \alpha\cdot\beta < \gamma\cdot\beta$.

5. (a) Let $\mathbf{F} : \mathbf{ON} \to \mathbf{ON}$ be a class function such that

   - $\mathbf{F}(\alpha^+) \geq \mathbf{F}(\alpha)$ for all $\alpha \in \mathbf{ON}$, and
   - if η is a limit ordinal, then $\mathbf{F}(\eta) = \bigcup_{\beta\in\eta} \mathbf{F}(\beta)$.

   (i) Show that $\mathbf{F}$ is increasing, i.e. if $\alpha \leq \beta \in \mathbf{ON}$ then $\mathbf{F}(\alpha) \leq \mathbf{F}(\beta)$.

   (ii) Show that $\mathbf{F}$ has a fixed point, i.e. there exists $\alpha \in \mathbf{ON}$ such that $\mathbf{F}(\alpha) = \alpha$. [(*Hint: consider* "$\mathbf{F}(\mathbf{F}(\ldots(0)\ldots))$".)]

   [(*Once you have seen the definition, note that the aleph function $\alpha \mapsto \aleph_\alpha$ has these properties, so there exists $\alpha \in \mathbf{ON}$ such that $\aleph_\alpha = \alpha$.*)]

   (b) Deduce that there exists a least ordinal ϵ_0 such that $\omega^{\epsilon_0} = \epsilon_0$.

6. Recall that we assume Foundation. Prove that there is no descending sequence $X_0 \ni X_1 \ni \ldots$ of sets, that is, there is no function f with domain $\mathbb{N}$ such that $f(n^+) \in f(n)$ for all $n \in \mathbb{N}$.

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7. Prove that the following are equivalent:

   (a) AC

   (b) Every surjection $f : X \to Y$ has a section, i.e. a function $g : Y \to X$ such that $f(g(y)) = y$ for all $y \in Y$.

   (c) The product of a family of non-empty sets is non-empty: if $f : I \to \mathbf{V}$ is a function and $\emptyset \notin \operatorname{ran}(f)$, then there exists a function $g : I \to \mathbf{V}$ such that $g(i) \in f(i)$ for all $i \in I$.

8. Let X be a *finite* set of disjoint non-empty sets. Show, without assuming AC, that there exists a set C such that $|C \cap a| = 1$ for all $a \in X$.

9. Assume AC. Show that partial orders can be linearised: if $\prec$ is a strict order on a set X, there exists a total order $<$ on X with $\prec \subseteq <$.

   [*Hint: Apply Zorn's Lemma.*]

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Section C

10. In this question, assume the axioms of ZF apart from Foundation. Recall the von Neumann cumulative hierarchy V_α. Let $\mathbf{W} := \{x : \exists \alpha \in \mathbf{ON}\ x \in V_\alpha\}$.
    (a) Show that $(V_\alpha)_{\alpha \in \mathbf{ON}}$ forms a chain of transitive sets, i.e. $\alpha \in \beta \Rightarrow V_\alpha \subseteq V_\beta$ and each V_β is transitive.
    (b) Show $x \in \mathbf{W} \Leftrightarrow x \subseteq \mathbf{W}$.
    (c) Show that Foundation is equivalent to the assertion $\mathbf{V} = \mathbf{W}$ that every set is an element of some V_α.
    (d) Deduce that Foundation can be reformulated as the following principle of "$\in$-induction": for any formula with parameters $\phi(x)$,

$$(\forall x\ (\forall y \in x\ \phi(y) \rightarrow \phi(x)) \rightarrow \forall x\ \phi(x)).$$

11. Show that Replacement is a strengthening of Comprehension. More precisely: assume Replacement, but do not assume Comprehension, and prove that Comprehension holds.

12. Assume AC. Let R be a commutative ring with identity $1 \neq 0$.
    (a) Prove that the union of a non-empty chain of proper ideals is a proper ideal.
    (b) Use Zorn's Lemma to prove that R has a maximal ideal.

13. Assume AC. Let $\prec$ be a well-order on $\mathbb{R}^{\mathbb{R}}$.

    Fix a function $f : \mathbb{R} \rightarrow \mathbb{R}$. Think of it as a (possibly highly discontinuous) function of time, such as inflation rate, Oxford precipitation, or atmospheric CO_2 concentrations.

    Given $t \in \mathbb{R}$, define the *Hardin-Taylor forecast* of f at t to be the $\prec$-least function $f_t^+ : \mathbb{R} \rightarrow \mathbb{R}$ among those which agree with f up until time t, i.e. $f_t^+ := \min_\prec \{g \in \mathbb{R}^{\mathbb{R}} : \forall s < t\ g(s) = f(s)\}$.

    Show that for all $t \in \mathbb{R}$ outside of a countable nowhere dense set, there is $\epsilon > 0$ such that the forecast at t is correct and remains correct for a further ϵ time, i.e. $f_t^+(s) = f(s)$ for all $s < t + \epsilon$.

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