Lie groups C3.5. HT 26 [P. Bousseau]

Lecture 13 [01/03/2026]

G connected compact Lie group $\rightsquigarrow$ $R(G) = \chi(R(G))$ representation ring

$T \subset G$ maximal torus $\rightsquigarrow$ $R(T) = \chi(R(T)) = \mathbb{Z}[t_1^{\pm}, \dots, t_r^{\pm}]$

$\dim T = \mathrm{rk}\, G = r$

Restriction map: $\underset{R(G)}{\underset{\parallel}{\chi(R(G))}} \longrightarrow \underset{R(T)^W}{\underset{\parallel}{\chi(R(T))^W}}$ W-invt part

W: Weyl group.

<u>Thm:</u> $\chi(R(G)) \xrightarrow{\sim} \chi(R(T))^W$

<u>Proof:</u> Injectivity? χ_V determined by $\chi_V|_T$

$h \in G$ $\exists t \in T, g \in G$ s.t. $h = gtg^{-1}$ so $\chi_V(h) = \chi_V|_T(t)$.

Surjectivity? Difficult. [No proof]. $\square$

<u>Ex:</u> $G = SU(2)$ $\quad T = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \middle| \theta \in \mathbb{R} \right\}$

$W = \mathbb{Z}/2\mathbb{Z}$ $\quad t \mapsto t^{-1}$

$$\chi(R(T))^W = \left\{ \sum_j m_j (t^j + t^{-j}) \right\} = \mathbb{Z}[t + t^{-1}]$$

$\exists V_n$ irred rep of $SU(2)$ $\Rightarrow$ Surjectivity: $R(G) = \mathbb{Z}[t + t^{-1}]$

$\Rightarrow$ All irred rep of $SU(2)$ are $\simeq V_n$!

Ex: $V_1 = \mathbb{C}^2 \quad \chi_{V_1} = t + t^{-1}$

$V_1 \otimes V_1 = ? \quad \chi_{V_1}\chi_{V_1} = (t + t^{-1})^2 = t^2 + 2 + t^{-2}$

$= \underbrace{(t^2 + 1 + t^{-2})}_{\substack{\chi_{V_2} \\ (x^2,\, xy,\, y^2)}} + \underbrace{1}_{\chi_{V_0}}$

so $V_1 \otimes V_1 \simeq V_2 \oplus V_0$.

Focus on $G = U(n) \quad W = S_n$

Lemma: $R(T)^W = \mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}]^{S_n} = \mathbb{Z}[\sigma_1, \dots, \sigma_n, \sigma_n^{-1}]$

$$\prod_{j=1}^{n} (x - t_j) = \sum_{k=0}^{n} (-1)^k \sigma_k x^{n-k} \qquad \sigma_k(t_1, \dots, t_n) = \sum_{1 \leq j_1 < \dots < j_k \leq n} t_{j_1} \cdots t_{j_k}$$

Ex: $\sigma_1 = t_1 + \dots + t_n \qquad \sigma_n = t_1 \cdots t_n$

Proof: $\mathbb{Z}[t_1, \dots, t_n]^{S_n} = \mathbb{Z}[\sigma_1, \dots, \sigma_n]$ from linear algebra.

$f \in \mathbb{Z}[t_1^{\pm}, \dots, t_n^{\pm}]^{S_n} \quad \exists N \geq 1$ s.t. $\sigma_n^N f \in \mathbb{Z}[t_1, \dots, t_n]^{S_n}$. $\square$

Lemma: $V = \mathbb{C}^n$ standard rep of $U(n) \Rightarrow \Lambda^k V$ irred rep of $U(n)$ with $\chi_{\Lambda^k V}(t) = \sigma_k(t) \quad \forall t \in T$.

Proof: $v_1, \dots, v_n$ unitary basis of V.

$\Rightarrow v_{j_1} \wedge \dots \wedge v_{j_k}$ basis for $\Lambda^k V$

$A \in U(n)$

$$\rho_{\Lambda^k V}(A)(v_{j_1} \wedge \dots \wedge v_{j_k}) = A v_{j_1} \wedge \dots \wedge A v_{j_k}$$

$\chi_{\Lambda^k V}(t) = \sigma_k$

Why $\Lambda^k V$ irred? $0 \neq U \subset \Lambda^k V$ $U(n)$-invt

U T-invt $\Rightarrow U = \bigoplus_i V_i$ as rep of T

$$\underset{\text{Rep of } T}{\Lambda^k V} = \bigoplus_{j_1 < \dots < j_k} \mathbb{C}\, t_1^{j_1} \dots t_k^{j_k}$$

Subset W-invt $\Rightarrow U = \Lambda^k V$. $\square$

<u>Prop</u>: $R(U(n)) = R(T)^W = \mathbb{Z}[\sigma_1, \dots, \sigma_n, \sigma_n^{-1}]$

<u>Proof</u>:

$\Lambda^k V$ irred rep with $\chi_{\Lambda^k V}|_T = \sigma_k$

$(\Lambda^n V)^*$ $\chi_{\Lambda^n V^*}|_T = \sigma_n^{-1}$. $\square$

<u>GOAL</u>: Weyl integration formula.

<u>Def</u>: $f : G \to \mathbb{C}$ continuous is a <u>class function</u> if $f(ghg^{-1}) = f(h)$ $\forall g, h \in G$

$\mathcal{E}(G) := \{\text{class functions on } G\}$

<u>Ex</u>: V $\mathbb{C}$-rep $\chi_V \in \mathcal{E}(G)$

<u>Pb</u>: How to express $\int_G f$ as $\int_T$?

See next time: Weyl integration formula.