C3.5 Lie Groups

Sheet 2 — HT26

Section A contains introductory questions. Section B contains material to test understanding of the course. Section C contains further questions which are optional. No answers should be submitted for marking.

Section A

1. The algebra of *quaternions* is defined as

$$\mathbb{H} = \{a + bi + cj + dk \ : \ a, b, c, d \in \mathbb{R}\}$$

where i, j, k satisfy the relations

$$ij = k = -ji \ \text{ and } \ i^2 = j^2 = k^2 = -1.$$

(a) Show that $jk = i = -kj$ and $ki = j = -ik$.

(b) Show that $\mathbb{H}$ may be identified with the algebra of matrices

$$\left\{ \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix} \ : \ z, w \in \mathbb{C} \right\}.$$

(c) If $q = a + bi + cj + dk \in \mathbb{H}$, we define the *quaternionic conjugate* to be

$$\bar{q} = a - bi - cj - dk.$$

(i) Show that $q\bar{q}$ is real and non-negative, so that the *norm* of q, which is the nonnegative real number $|q|$ such that $|q|^2 = q\bar{q}$, is well-defined.

(ii) Deduce that $q \in \mathbb{H} \setminus \{0\}$ has a multiplicative inverse $q^{-1} = \frac{\bar{q}}{|q|^2}$.

(d) (i) Show that, for $q_1, q_2 \in \mathbb{H}$ and $q \in \mathbb{H} \setminus \{0\}$,

$$|q_1 q_2| = |q_1| \cdot |q_2| \quad \text{and} \quad |q^{-1}| = |q|^{-1}.$$

(ii) Viewing $\mathbb{H}$ as a real 4-dimensional vector space, check that $|q|$ is the usual norm on $\mathbb{R}^4$.

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Solution:

(a) We have $jk = -i^2jk = -ik^2 = i = -k^2i = kji^2 = -kj$ and
$ki = -kij^2 = -k^2j = j = -jk^2 = -ik$.

(b) Let

$$A = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} : z, w \in \mathbb{C} \right\}.$$

An $\mathbb{R}$-algebra isomorphism $\theta : \mathbb{H} \to A$ is given by

$$a + bi + cj + dk \mapsto \begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}.$$

By inspection θ is compatible with the relations defining $\mathbb{H}$ and is $\mathbb{R}$-linear, so is a genuine homomorphism of $\mathbb{R}$-algebras that is also clearly bijective.

(c) (i) If $q = a + bi + cj + dk$ then

$$q\overline{q} = (a + bi + cj + dk)(a - bi - cj - dk) = a^2 + b^2 + c^2 + d^2 \in \mathbb{R}^{\geq 0}.$$

(ii) With $q \neq 0$ and $|q| = \sqrt{q\overline{q}}$ we have $q\overline{q}/|q|^2 = 1$ so $q^{-1} = \overline{q}/|q|^2$.

(d) (i) We have (by a quick calculation) $\overline{q_1q_2} = \overline{q_2} \cdot \overline{q_1}$ and $q\overline{q} = \overline{q}q$ then

$$|q_1q_2|^2 = q_1 \cdot q_2 \cdot \overline{q_2} \cdot \overline{q_1} = q_1|q_2|^2\overline{q_1} = |q_1|^2|q_2|^2.$$

Taking square roots yields $|q_1q_2| = |q_1||q_2|$. Taking $q_1 = q$ and $q_2 = q^{-1}$ gives $|q||q^{-1}| = |1| = 1$, hence $|q^{-1}| = |q|^{-1}$.
Alternatively, by direct calculation

$$|q|^2 = \det \theta(q),$$

so the multiplicativity of the quaternionic norm follows from the multiplicativity of the determinant.

(ii) This is immediate from the earlier calculation that $|q|^2 = a^2 + b^2 + c^2 + d^2$ for $q = a + bi + cj + dk$.

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2. Calculate the Lie algebras of the following Lie groups. (Note that this means finding both the vector space and the Lie bracket.)

   (a) The isometric transformations of $\mathbb{R}^2$ of the form $x \mapsto Ax + b$.

   (b) The non-zero quaternions $\mathbb{H}^*$.

   (c) The unit quaternions $\{q \in \mathbb{H} : |q| = 1\}$.

   (d) The group of Möbius transformations of the form

$$z \mapsto \frac{az+b}{cz+d}$$

   where $a, b, c, d \in \mathbb{R}$ and $ad - bc > 0$.

   [*Hint: It may be helpful to consider a homomorphism from a subgroup of* $GL(2, \mathbb{R})$ *to this group.*]

Solution:

(a) The group G of isometric transformations of $\mathbb{R}^2$ of the form $x \mapsto Ax + b$ can be identified with the subgroup of $GL(3, \mathbb{R})$ consisting of matrices of the form

$$\begin{pmatrix} A_{11} & A_{12} & b_1 \\ A_{21} & A_{22} & b_2 \\ 0 & 0 & 1 \end{pmatrix}$$

where $A \in \mathrm{O}(2)$. Thus the Lie algebra of G is a subalgebra of the Lie algebra of $GL(3, \mathbb{R})$ with Lie bracket given by commutator of matrices. $O(2)$ has Lie algebra the skew-symmetric 2×2 matrices, so a basis for the Lie algebra of G is

$$X = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; Z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and the Lie brackets are:

$$[X, Y] = 0, \; [Y, Z] = -X, \; [Z, X] = -Y.$$

(b) The nonzero quaternions form an open subset in $\mathbb{R}^4$ so the tangent space at the identity is $\mathbb{R}^4 = \mathbb{H}$. By left multiplication the nonzero quaternions form a subgroup of $GL(4, \mathbb{R})$ with the Lie bracket again the commutator. So the Lie algebra is spanned by $1, i, j, k$ and $[1, q] = 0$ for all $q \in \mathbb{H}$. The remaining Lie brackets are determined by

$$[i, j] = 2k, \; [j, k] = 2i, \; [k, i] = 2j.$$

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(c) The unit quaternions form the unit sphere in $\mathbb{R}^4$ whose tangent space at 1 is the orthogonal complement of $\mathbb{R} \subseteq \mathbb{H}$, namely the imaginary quaternions. The Lie brackets are as above.

(d) The composition of this group G of Möbius transformations is achieved by multiplying the corresponding 2×2 matrices. This means there is a surjective homomorphism from the subgroup of $GL(2, \mathbb{R})$ consisting of matrices of strictly positive determinant to G and a corresponding surjective map from the Lie algebra of $GL(2, \mathbb{R})$ to the Lie algebra of G. The Lie bracket for the matrix group is again commutator of matrices. The scalar matrices in $GL(2, \mathbb{R})$ give the trivial Möbius transformation, so the Lie algebra homomorphism maps the 3-dimensional Lie algebra of $SL(2, \mathbb{R})$, which consists of the trace zero 2×2 real matrices, surjectively to the 3-dimensional Lie algebra of G. This is therefore an isomorphism of Lie algebras.

Take a basis

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and the Lie brackets are

$$[X, Y] = Z, \ [Y, Z] = 2Y, \ [Z, X] = 2X.$$

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Section B

3. Define the Lie group (called the *compact symplectic group*) by

$$\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n, \mathbb{H}) : \overline{A^{\mathrm{T}}}A = I\},$$

where $\overline{A^{\mathrm{T}}}$ denotes the quaternionic conjugate transpose of A (ie the (i, j) entry of $\overline{A^{\mathrm{T}}}$ is the quaternionic conjugate of the (j, i) entry of A).

(a) Find the dimension of $\mathrm{Sp}(n)$ and the Lie algebra $\mathfrak{sp}(n)$ of $\mathrm{Sp}(n)$.

(b) Show that

$$\mathrm{Sp}(1) = \mathrm{SU}(2)$$

and that $\mathrm{Sp}(1)$ is topologically the 3-sphere.

(c) For $q \in \mathbb{H} \setminus \{0\}$ define

$$\mathcal{A}_q : \mathbb{H} \to \mathbb{H}, \quad p \mapsto qpq^{-1}.$$

Show that $\mathcal{A}_q$ is an orthogonal map (viewing $\mathbb{H}$ as $\mathbb{R}^4$).

(d) By considering the orthogonal complement of $\mathbb{R} = \mathbb{R} \cdot 1 \subset \mathbb{H}$, deduce that $\mathrm{SU}(2) \cong \mathrm{Sp}(1) \subset \mathbb{H} \setminus \{0\}$ acts on $\mathbb{R}^3$ by rotations.

(e) (Optional) Explain briefly why this gives a homomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2) \to \mathrm{SO}(3)$ with kernel $\{\pm 1\}$.

Solution:

(a) Using the regular value theorem, we see that $\mathrm{Sp}(n)$ is a manifold with

$$\mathfrak{sp}(n) = T_I\mathrm{Sp}(n) = \{B \in M_n(\mathbb{H}) \, : \, \overline{B^{\mathrm{T}}} + B = 0\}.$$

This is therefore the Lie algebra with the matrix commutator as the Lie bracket. Its dimension is $4 * \frac{1}{2}n(n-1) = 2n(n-1)$ for the off-diagonal entries plus $3 * n$ for the diagonal entries (which are purely imaginary), which is a total of $2n^2 + n = n(2n+1)$, which is then the dimension of $\mathrm{Sp}(n)$.

(b) $\mathrm{Sp}(1)$ is precisely the set of all quaternions q with $|q|^2 = 1$. Identifying $\mathbb{H} \equiv \mathbb{R}^4$ induces an identification $\mathrm{Sp}(1) \equiv S^3$. Identifying $\mathbb{H} \equiv A$ via θ from Question 1 induces the identification $\mathrm{Sp}(1)$ with $\mathrm{SU}(2)$.

(c) Observe that if $v = a + bi + cj + dk$ and $w = a' + b'i + c'j + d'k$ then by direct calculation

$$\langle v, w\rangle = \mathrm{Re}(v\overline{w}) = \frac{1}{2}(v\overline{w} + \overline{v\overline{w}}) = \frac{1}{2}(v\overline{w} + w\overline{v}).$$

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Without loss of generality we may assume q has unit norm, so $q^{-1} = \overline{q}$. Then

$$\begin{aligned}\langle \mathcal{A}_q(v), \mathcal{A}_q(w)\rangle &= \langle qvq^{-1}, qwq^{-1}\rangle \\ &= \frac{1}{2}\left(qv\overline{q}\cdot\overline{qw\overline{q}} + qw\overline{q}\cdot\overline{qv\overline{q}}\right) \\ &= \frac{1}{2}\cdot q\,(v\overline{w} + w\overline{v})\,\overline{q} \\ &= \frac{1}{2}(v\overline{w} + w\overline{v}) = \langle v, w\rangle,\end{aligned}$$

since $\mathcal{A}_q|_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}}$. Therefore $\mathcal{A}_q$ is an orthogonal map on $\mathbb{R}^4$.

(d) Let $V = \mathrm{Im}(\mathbb{H}) \cong \mathbb{R}^3$ be the orthogonal complement of $\mathbb{R} \subset \mathbb{H} = \mathbb{R}^4$. Since $\mathrm{Sp}(1) \subset \mathbb{H}^*$ acts by orthogonal transformations on $\mathbb{R}^4$ and restricts to the identity on $\mathbb{R}$ then $\mathcal{A}_q(V) = V$ for all $q \in \mathrm{Sp}(1)$.

To show $\mathrm{Sp}(1)$ acts on $\mathbb{R}^3$ by rotations, we consider the composition

$$S^3 = \mathrm{Sp}(1) \xrightarrow{\quad\mathcal{A}\quad} \mathrm{O}(3) \xrightarrow{\quad\det\quad} \{\pm 1\}.$$

This gives a continuous map to a discrete space; as $\mathrm{Sp}(1)$ is connected this map is necessarily constant. But $1 \in \mathrm{Sp}(1)$ and $\det(\mathcal{A}_1) = 1$, so $\mathcal{A}_q \in \mathrm{SO}(3)$ for all $q \in \mathrm{Sp}(1)$. In other words $\mathrm{Sp}(1)$ acts on $\mathbb{R}^3$ by rotations.

(e) The homomorphism in question is given by $\mathcal{A}$ (which is in fact a homomorphism of Lie groups). The elements ± 1 lie in the kernel of this map; we will show these are the only elements. We will do this by showing that $\mathcal{A}$ is a non-trivial covering map then appealing to the fact that the fundamental group $\pi_1(SO(3)) = \mathbb{Z}/2$.

We first compute the derivative at 1 of $\mathcal{A}$, viewed as a map $\mathbb{R}^4 \to M_4(\mathbb{R})$ (where $SO(3) \hookrightarrow M_4(\mathbb{R})$ via $A \mapsto \begin{pmatrix} 1 & \\ & A\end{pmatrix}$). Take $q, h \in \mathbb{H}$ with $|h| < 1$. We may expand $(1+h)^{-1}$ as an infinite series:

$$(1+h)^{-1} = \sum_{n=0}^{\infty}(-1)^n h^n.$$

Then

$$\begin{aligned}(\mathcal{A}_{1+h} - \mathcal{A}_1)(q) &= (1+h)q(1+h)^{-1} - q \\ &= (1+h)q(1 - h + o(h)) - q \\ &= hq - qh + o(h) = [h, q] + o(h).\end{aligned}$$

It follows that

$$(d\mathcal{A})_1 : \mathfrak{su}(2) \cong T_1 Sp(1) \to \mathfrak{so}(3), \qquad h \mapsto [h, -].$$

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It can easily be shown that this map is an isomorphism of Lie algebras (the Lie bracket of S^3 is given by the cross product on $\mathbb{R}^3$ - see Question 5). This implies that $\mathcal{A} : Sp(1) \to SO(3)$ is a covering map, which is non-trivial since $\mathcal{A}$ has non-trivial kernel.

From algebraic topology there exists a homeomorphism $\mathrm{SO}(3) \cong \mathbb{RP}^3$. But S^3 is the universal cover of $\mathbb{RP}^3$ via the obvious two-to-one quotient map, so S^3 is also the universal cover of SO(3). In particular this implies that $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}/2$ and that any non-trivial covering of SO(3) is equivalent to the covering $S^3 \to \mathbb{RP}^3 \cong \mathrm{SO}(3)$. Therefore $\mathcal{A} : \mathrm{Sp}(1) \cong \mathrm{SU}(2) \to \mathrm{SO}(3)$ is a double covering and induces an isomorphism $\mathrm{Sp}(1)/\{\pm 1\} \cong \mathrm{SO}(3)$.

4. Check these properties of $\exp : \mathfrak{g} = \mathrm{Lie}(G) \to G$ for a Lie group G with identity e.
   (a) $\mathrm{Image}(\exp) \subseteq G_0$ where G_0 = connected component of $e \in G$.
   (b) $\exp((s+t)X) = \exp(sX)\exp(tX)$ for all $s, t \in \mathbb{R}$ and $X \in \mathfrak{g}$.
   (c) $\big(\exp(X)\big)^{-1} = \exp(-X)$ for all $X \in \mathfrak{g}$.
   (d) If $g = \exp(X)$ then it has an n-th root.
   (e) $\exp : \mathfrak{sl}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$ is not surjective for $n \geq 2$.

Solution:

(a) Recall that if $X \in \mathfrak{g}$ then $\exp(X) = \alpha^X(1)$, where α^X is the 1-parameter subgroup with tangent vector X at the identity. But $t \to \alpha^X(t)$ is a path in G with $\alpha^X(0) = e$, so $\exp(X)$ must lie in the same path component as e.

(b) By definition $\exp((s+t)X) = \alpha^{(s+t)X}(1)$. For any $\lambda \in \mathbb{R}$ we have $\alpha^{\lambda X}(u) = \alpha^X(\lambda u)$ as both curves have tangent vector λX at the identity. Therefore $\exp((s+t)X) = \alpha^X(s+t) = \alpha^X(s)\alpha^X(t) = \exp(sX)\exp(tX)$.

(c) Taking $s = -t = 1$ gives $\exp(X)\exp(-X) = \exp(0) = e$, hence $\exp(X)^{-1} = \exp(-X)$.

(d) If $g = \exp(X)$ then an n-th root of g is given by $\exp(X/n)$.

(e) Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), C = \left(\begin{smallmatrix} -2 & 0 \\ 0 & -1/2 \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{R})$ and suppose for a contradiction that $A^2 = C$ (for some choice of $a, \ldots, d$). Then $b(a+d) = 0$ and $a^2 + bc = -2$. This forces $b \neq 0$, so $a = -d$ and $1 = ad - bc = -(a^2 + bc) = 2$, contradiction. Therefore C has no square root so cannot lie in the image of exp. Then by embedding A in $\mathrm{SL}(n, \mathbb{R})$ in the obvious way for any $n \geq 2$ gives the result.

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5. (a) Prove directly that ad is a Lie algebra homomorphism from $\mathrm{ad}(X)(Z) = [X, Z]$ for X, Z in the Lie algebra.

(b) Show that

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

is a basis for the Lie algebra $\mathfrak{so}(3) \subset M_3(\mathbb{R})$ of SO(3).

(c) By computing $[X_i, X_j]$ for $i, j = 1, 2, 3$, show that if e_1, e_2, e_3 are the standard basis vectors on $\mathbb{R}^3$ and $\times$ is the cross product on $\mathbb{R}^3$, then

$$F : \mathfrak{so}(3) \to (\mathbb{R}^3, \times), \qquad X_i \mapsto e_i$$

is a Lie algebra isomorphism.

(d) Via F in the previous part we identify $\mathrm{End}(\mathfrak{so}(3))$ with 3×3 matrices. Compute the matrices $\mathrm{ad}(X_i)$.

(e) By computing $\kappa(X_i, X_j)$ for $i, j = 1, 2, 3$ show that the *Killing form*

$$\kappa(X, Y) = \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) \in \mathbb{R}$$

is a negative definite scalar product on $\mathfrak{so}(3)$.

Solution:

(a) Let $X, Y, Z \in \mathfrak{g}$. From the given expression for $\mathrm{ad}(X)$ we already have that ad is linear, so it remains to show that

$$\mathrm{ad}([X, Y])(Z) = [\mathrm{ad}(X), \mathrm{ad}(Y)](Z).$$

But by antisymmetry and the Jacobi identity

$$\begin{aligned} [\mathrm{ad}(X), \mathrm{ad}(Y)] \cdot Z &= (\mathrm{ad}(X) \circ \mathrm{ad}(Y) - \mathrm{ad}(Y) \circ \mathrm{ad}(X)) \cdot Z \\ &= \mathrm{ad}(X)([Y, Z]) - \mathrm{ad}(Y)([X, Z]) \\ &= [X, [Y, Z]] - [Y, [X, Z]] \\ &= [X, [Y, Z]] + [Y, [Z, X]] \\ &= -[Z, [X, Y]] \\ &= [[X, Y], Z] \\ &= \mathrm{ad}([X, Y])(Z). \end{aligned}$$

Therefore ad is a Lie algebra homomorphism.

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(b) The X_i are linearly independent by inspection. Any skew-symmetric 3×3 real matrix must have zeros on the diagonal, and is uniquely determined by the entries a_{ij} with $i < j$. Therefore the X_i span $\mathfrak{so}(3)$.

(c) The Lie bracket on $\mathfrak{so}(3)$ inherited from the Lie group SO(3) coincides with the matrix commutator. Quick computations then give

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2.$$

Then F is isomorphism of Lie algebras, as

$$e_1 \times e_2 = e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = e_2.$$

(d) With respect to the ordered basis (X_1, X_2, X_3), by inspection

$$\mathrm{ad}(X_1) = -X_3, \quad \mathrm{ad}(X_2) = -X_2, \quad \mathrm{ad}(X_3) = -X_1.$$

(e) We have for all i and j

$$\kappa(X_i, X_j) = -2\delta_{ij}.$$

By the linearity of trace and ad the Killing form is bilinear, and is symmetric as $\mathrm{tr}(AB) = \mathrm{tr}(BA)$. Given $X \in \mathfrak{so}(3)$, expanding out X as a linear combination of the X_i gives $\kappa(X, X) \leq 0$, with equality if and only if $X = 0$. Therefore the Killing form is a negative definite scalar product on $\mathfrak{so}(3)$.

6. (a) Show that for a matrix group G, we have $\exp(gXg^{-1}) = g \exp(X) g^{-1}$ for all $g \in G$ and $X \in \mathfrak{g}$.

(b) Consider the subgroup T of the unitary group $\mathrm{U}(n)$ consisting of diagonal matrices. Show that T is a torus T^n and that T lies in the image of the exponential map $\exp : \mathfrak{u}(n) \to \mathrm{U}(n)$.

(c) Deduce that $\exp : \mathfrak{u}(n) \to \mathrm{U}(n)$ is surjective.

Solution:

(a) Fix $g \in G$ and consider the Lie group endomorphism $C_g : G \to G,\ h \mapsto ghg^{-1}$. By definition we have $\mathrm{Ad}_g = (dC_g)_I : \mathfrak{g} \to \mathfrak{g}$. By the naturality of the exponential map the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\mathrm{Ad}_g} & \mathfrak{g} \\ \downarrow{\scriptstyle \exp} & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow{C_g} & G \end{array}$$

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Identifying $\mathfrak{g}$ as a matrix Lie algebra, we have for $X \in \mathfrak{g}$ the identity

$$\mathrm{Ad}(g)(X) = gXg^{-1}$$

since the map $A \mapsto PAP^{-1}$ on matrices is linear. The equality $\exp(gXg^{-1}) = g\exp(X)g^{-1}$ follows.

(b) If $A \in T$, the equality $\overline{A}^T A = I$ implies that all of the diagonal entries of A are complex numbers of unit norm, so

$$T = \left\{ \mathrm{diag}(e^{it_1}, \ldots, e^{it_n}) : t_i \in \mathbb{R} \right\} \cong (S^1)^n.$$

The exponential map on $\mathfrak{u}(n)$ is given by the usual matrix exponential $A \mapsto \sum_{n=0}^{\infty} A^n/n!$. If $A \in T$, the equality $\overline{A}^T A = I$ implies that all of the diagonal entries of A are complex numbers of unit norm, so

$$T = \left\{ \mathrm{diag}(e^{it_1}, \ldots, e^{it_n}) : t_i \in \mathbb{R} \right\} \cong (S^1)^n.$$

The exponential map on $\mathfrak{u}(n)$ is given by the usual matrix exponential $A \mapsto \sum_{n=0}^{\infty} A^n/n!$.(In $GL(k, \mathbb{R})$, the curve $t \mapsto \sum_{n=0}^{\infty}(tB)^n/n!$ is a smooth curve in $GL(k, \mathbb{R})$ with derivative B at $t = 0$, so by uniqueness of integral curves this is the unique integral curve through I with tangent vector B in $GL(k, \mathbb{R})$. Any matrix Lie group is a Lie subgroup of $GL(k, \mathbb{R})$ for some k. Thus for any matrix Lie group, the Lie group and matrix exponentials coincide.)

As

$$\mathrm{diag}(e^{it_1}, \ldots, e^{it_n}) = \exp(\mathrm{diag}(it_1, \ldots, it_n))$$

then T lies in the image of $\exp : \mathfrak{u}(n) \to U(n)$.

(c) Given $A \in U(n)$, there exists a diagonal matrix D and a unitary matrix P with $A = PDP^{-1}$. Then $D \in T$ so is equal to $\exp(B)$ for some $B \in \mathfrak{u}(n)$. Then (using the first part of this question) $A = \exp(PBP^{-1})$. As PBP^{-1} is skew-Hermitian then A lies in the image of $\exp : \mathfrak{u}(n) \to U(n)$.

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Section C

7. The 3-dimensional Heisenberg group G consists of matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$.

(a) Show that the Lie algebra of G consists of matrices

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Calculate the exponential map for G.

(c) Is the exponential map surjective in this case?

8. (a) If $A \in \mathrm{GL}(n, \mathbb{C})$ is diagonalizable, show that $A = \exp B$ for a complex matrix B.

(b) Let

$$A = \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

with $\lambda \neq 0 \in \mathbb{C}$. Show, by writing this in the form $\lambda(I + N)$, that in this case too there exists B such that $A = \exp B$.

(c) The Jordan normal form states that any complex $n \times n$ matrix is conjugate to a matrix with blocks of the above form down the diagonal. Deduce that the exponential map for the Lie group $\mathrm{GL}(n, \mathbb{C})$ is surjective.

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