

## B3.3 Algebraic Curves. Last three lectures

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In the last three lectures, we will cover the following topics:

- Meromorphic functions
- Divisors and ramification divisors. The degree of a non constant map between Riemann surfaces.
- The Riemann-Hurwitz formula
- The genus-degree formula.
- The Riemann-Roch formula.

# Meromorphic functions

Let  $X$  be a Riemann surface.

A *meromorphic function* on  $X$  is a map  $f : X \rightarrow \mathbb{C} \cup \{\infty\}$ , such that  $\phi^{-1} \circ f : V \rightarrow \mathbb{C} \cup \{\infty\}$  is a meromorphic function for any chart  $\phi : U \rightarrow V \subseteq \mathbb{C}$  on  $X$ .

If  $f : X \rightarrow \mathbb{C} \cup \{\infty\}$  is a meromorphic function, the map  $X \rightarrow \mathbb{P}^1(\mathbb{C})$  such that

- $p \in X \mapsto [f(p), 1]$  if  $f(p) \neq \infty$
- $p \in X \mapsto [1, 1/f(p)]$  if  $f(p) \neq 0$

is a holomorphic map between Riemann surfaces.

Conversely, if  $g : X \rightarrow \mathbb{P}^1(\mathbb{C})$  is a map of Riemann surfaces such that  $g(p) = [g_0(p), g_1(p)]$  for all  $p \in X$ , then the function  $X \rightarrow \mathbb{C} \cup \{\infty\}$  such that  $g(p) = g_0(p)/g_1(p)$  is a meromorphic function.

These two recipes are inverse to each other, so that a meromorphic function can be described as a holomorphic map  $X \rightarrow \mathbb{P}^1(\mathbb{C})$ .

**Example.** Let  $C \subseteq \mathbb{P}^2(\mathbb{C})$  be a non singular algebraic curve, such that  $[0, 0, 1] \notin C$ .

Then the map  $C \rightarrow \mathbb{C} \cup \{\infty\}$  such that  $[X, Y, Z] \mapsto X/Y$  for all  $[X, Y, Z] \in C$  is a meromorphic function on  $C$ .

This follows from our description of the coordinate charts of  $C$  and from the fact that the composition of two complex differentiable functions (in several variables) is also complex differentiable.

We shall now turn to a study of the local structure of non constant holomorphic maps between Riemann surfaces.

## Lemma 4.4

Let  $U \subseteq \mathbb{C}$  be an open neighbourhood of 0 in  $\mathbb{C}$ . Let  $f : U \rightarrow \mathbb{C}$  be a holomorphic function such that  $f(0) = 0$  and such that  $f$  is not constant in some neighbourhood of 0.

Then there are open neighbourhoods  $U_0$  and  $V_0$  of 0 and a bijective holomorphic map  $\phi : U_0 \rightarrow V_0$ , such that

- $U_0 \subseteq U$ ;
- $\phi(0) = 0$  and  $\phi^{-1}$  is also holomorphic;
- we have  $f(\phi^{-1}(w)) = w^n$  for some  $n \geq 1$ .

**Proof of the lemma.** In a sufficiently small open disk  $B(r, 0)$ , write  $f(z) = z^n g(z)$ , where  $g(z)$  is holomorphic and  $g(0) \neq 0$ . Decreasing  $r$  is necessary, suppose that  $g(B(r, 0)) \subseteq B(|g(0)|, 0)$ .

The open disk  $B(|g(0)|, 0)$  is simply connected and does not contain 0, so there is branch of the logarithm  $\ell : B(|g(0)|, 0) \rightarrow \mathbb{C}$ .

Let  $h(z) := ze^{\frac{1}{n}\ell(g(z))}$  for all  $z \in B(r, 0)$ . Then

$$(h(z))^n = f(z)$$

and

$$h'(0) = e^{\frac{1}{n}\ell(g(0))} \neq 0.$$

So  $h$  is locally invertible around 0 by the holomorphic inverse function theorem. Let  $U_0 \subseteq B(r, 0)$  be an open neighbourhood of 0 such that  $h(U_0)$  is open and such that the induced map  $U_0 \rightarrow h(U_0)$  is biholomorphic. Define  $V_0 := h(U_0)$  and  $\phi := h|_{U_0}$ .

Then  $f(\phi^{-1}(w)) = w^n$  for all  $w \in V_0$ .  $\square$

## Proposition 4.5

Let  $f : X \rightarrow Y$  be a non constant holomorphic map between compact and connected Riemann surfaces.

Then  $f$  is open and surjective, and for any  $q \in Y$ , the set  $f^{-1}(\{q\})$  is finite.

Furthermore, for any  $q \in Y$ , there are

- a coordinate chart  $\phi : U \rightarrow V \subseteq \mathbb{C}$  around  $q$  such that  $\phi(q) = 0$ ;
- coordinate charts  $\phi_p : U_p \rightarrow V_p \subseteq \mathbb{C}$  around each  $p \in f^{-1}(\{q\})$  such that  $\phi_p(p) = 0$ ;
- integers  $n(p) \geq 1$  for each  $p \in f^{-1}(\{q\})$ ,

such that

- $f^{-1}(U) = \cup_{p \in f^{-1}(\{q\})} U_p$ ;
- $U_p \cap U_{p'} = \emptyset$  if  $p \neq p'$ ;
- $\phi \circ f \circ \phi_p^{-1}(w) = w^{n(p)}$  for all  $w \in V_p$ .

Finally, we may assume that  $V = B(1, 0)$  and that  $V_p = B(1, 0)$  for all  $p \in f^{-1}(\{q\})$ .

**Proof of the Proposition.** The map  $f$  is open because non constant holomorphic maps from connected open subsets of  $\mathbb{C}$  to  $\mathbb{C}$  are open. The set  $f^{-1}(\{q\})$  is finite by the identity principle and by the compactness of  $X$ .

Choose any coordinate chart  $\phi' : U' \rightarrow V' \subseteq \mathbb{C}$  around  $q$  such that  $\phi'(q) = 0$ . Choose coordinate charts  $\phi'_p : U'_p \rightarrow V'_p \subseteq \mathbb{C}$  around each  $p \in f^{-1}(\{q\})$  such that  $\phi'_p(p) = 0$ ,  $\phi'(U'_p) \subseteq U'$ , and such that  $\phi' \circ f \circ (\phi'_p)^{-1}(w) = w^{n(p)}$  for some  $n(p) \geq 1$ . This is possible by the previous lemma. Since  $f^{-1}(\{q\})$  is finite, we may suppose that the  $V'_p$  are pairwise disjoint.

Now let  $S := X \setminus \bigcup_{p \in f^{-1}(\{q\})} U'_p$ . We have by construction

$$f^{-1}(Y \setminus f(S)) \subseteq \bigcup_{p \in f^{-1}(\{q\})} U'_p$$

and since  $q \notin f(S)$ , we have

$$f^{-1}(q) \subseteq f^{-1}(Y \setminus f(S)).$$

Note that  $f(S)$  is compact since  $f$  is continuous, and thus closed.

Now let  $U = U' \cap (Y \setminus f(S))$  and  $U_p = f^{-1}(U) \cap U'_p$  for all  $p \in f^{-1}(\{q\})$ , and define  $\phi : U \rightarrow V$  and  $\phi_p : U_p \rightarrow V_p$  by restricting  $\phi'$  and  $\phi'_p$ , respectively.

To obtain to last statement, let  $p \in f^{-1}(\{q\})$  and consider the map  $\phi \circ \phi_p^{-1} : V_p \rightarrow V$ .

By construction, both  $V$  and  $V_p$  contain 0, and we have

$$(\phi \circ \phi_p^{-1})(w) = w^n$$

for all  $w \in V_p$ . Choose a disk  $B(\rho, 0) \subseteq V$ . The set of  $n$ -th roots of  $B(\rho, 0)$  in  $\mathbb{C}$  is  $B(\sqrt[n]{\rho}, 0)$ , so if we choose  $\rho$  sufficiently small, we may assume that

$$(\phi \circ \phi_p^{-1})^{-1}(B(\rho, 0)) = B(\sqrt[n]{\rho}, 0) \subseteq V_p.$$

So we may replace  $V$  by  $B(\rho, 0)$  and  $V_p$  by  $B(\sqrt[n]{\rho}, 0)$ . A simple rescaling now gives the required form for  $V$  and  $V_p$ .  $\square$

The last proposition has many applications. We first need a definition.

### Definition 4.6

Let  $f : X \rightarrow Y$  be a holomorphic map between Riemann surfaces.

Let  $p \in X$  and let  $q := f(p)$ .

Let  $\psi : U_p \rightarrow V_p$  and  $\phi : U \rightarrow V$  be coordinate charts around  $p$  and  $q$  respectively. Suppose that  $\psi(p) = 0$  and that  $\phi(q) = 0$ .

The ramification degree  $\text{RD}(f, p)$  of  $f$  at  $p$  is the natural number

$$1 + \max\{n \mid \frac{d^k f}{dw^k}(\phi \circ f \circ \psi^{-1})(\psi(p)) = 0 \quad \forall k \in \{0, \dots, n\}\}$$

Note that the ramification degree does not depend on the choice of coordinate charts.

Also, if  $X$  is compact, then for all but a finite number of  $p \in X$ , we have  $\text{RD}(f, p) = 1$ . This follows from the identity principle.

## Proposition 4.7

Let  $f : X \rightarrow Y$  be a non constant holomorphic map between compact and connected Riemann surfaces.

Then the function

$$y \in Y \mapsto \sum_{x \in f^{-1}(\{y\})} \text{RD}(f, x)$$

is constant on  $Y$ .

The constant value of the function  $\sum_{x \in f^{-1}(\{y\})} \text{RD}(f, x)$  on  $Y$  is called the *degree* of  $f$ .

**Proof.** Consider the situation of Proposition 4.5, with  $V = V_p = B(1, 0)$  (for all  $p$ ).

Then we have

$$\sum_{p \in f^{-1}(\{q\})} \text{RD}(f, p) = \sum_{p \in f^{-1}(\{q\})} n(p).$$

If  $x \in U \setminus \{q\}$  and  $y \in f^{-1}(\{x\})$ , then  $\text{RD}(f, x) = 1$  and  $x$  has  $n(p)$  preimages in  $U_p$ .

Hence, if  $x \in U \setminus \{q\}$ , we have

$$\sum_{x \in f^{-1}(\{y\})} \text{RD}(f, x) = \sum_{p \in f^{-1}(\{q\})} \sum_{x \in U_p \cap f^{-1}(\{x\})} \text{RD}(f, x) = \sum_{p \in f^{-1}(\{q\})} n(p)$$

We conclude that the function under consideration is locally constant on  $Y$ , and hence constant, since  $Y$  is connected.  $\square$

## Definition 4.8

Let  $X$  be a Riemann surface. A function  $D : X \rightarrow \mathbb{Z}$  which vanishes for all but finitely many  $x \in X$  is called a *divisor* (of  $X$ ).

The quantity  $\sum_{x \in X} D(x)$  is called the *degree* of  $D$ .

**Examples.** Let  $f : X \rightarrow Y$  be a non constant holomorphic map between compact and connected Riemann surfaces. Let  $q \in Y$ .

Then the function  $D : X \rightarrow \mathbb{Z}$  such that  $D(p) = 0$  if  $p \notin f^{-1}(\{q\})$ , and such that  $D(p) = \text{RD}(f, p)$  otherwise, is called the *fibre divisor* of  $f$  above  $q$  (this terminology is not quite standard).

For instance, if  $Y = \mathbb{P}^1(\mathbb{C})$ , then the fibre divisor above  $\infty$  (resp.  $0$ ) is called the divisor of poles (resp. zeroes) of  $f$ . In that case, the divisor (divisor of zeroes) - (divisor of poles) is denoted by  $(f)$ . Note that by Proposition 4.7, we have  $\deg((f)) = 0$ .

Let  $X$  be a Riemann surface.

### Definition 4.9

A meromorphic differential  $\omega$  on  $X$  is an assignment of a meromorphic function  $\omega_V : V \rightarrow \mathbb{C} \cup \{\infty\}$  to any coordinate chart  $\phi : U \rightarrow V$  of  $X$ , subject to the following conditions. If  $\phi_1 : U_1 \rightarrow V_1$ ,  $\phi_2 : U_2 \rightarrow V_2$  are any two coordinate charts, then we have

$$\omega_{V_1}|_{\phi_1(U_1 \cap U_2)}(w) = \frac{d}{dw}(\phi_2 \circ \phi_1^{-1})(w) \cdot (\omega_{V_2} \circ \phi_2 \circ \phi_1^{-1})(w)$$

for all  $w \in \phi_1(U_1 \cap U_2)$ .

Suppose that  $X$  is connected.

If  $\omega_1$  and  $\omega_2$  are two non-zero meromorphic differentials, then the function  $\omega_1/\omega_2 : X \rightarrow \mathbb{C} \cup \{\infty\}$  such that  $(\omega_1/\omega_2)(x) = \omega_{1,V}(\phi(x))/\omega_{2,V}(\phi(x))$  for any coordinate chart  $\phi : U \rightarrow V$  is well-defined and is a meromorphic function on  $X$ .

Conversely, if  $\omega$  is a meromorphic differential on  $X$ , and  $f : X \rightarrow \mathbb{C} \cup \{\infty\}$  is a meromorphic function, then we can define a meromorphic differential  $f\omega$  by associating with any coordinate chart  $\phi : U \rightarrow V$  the meromorphic differential  $(f \circ \phi^{-1}) \cdot \omega_V$ .

The set of meromorphic differentials is thus naturally a one-dimensional vector space over the field of meromorphic functions.

If  $f$  is a meromorphic function on  $X$ , then the assignment of the function  $\frac{d}{dw}(f \circ \phi^{-1})(w)$  to any coordinate chart  $\phi : U \rightarrow V$  is a meromorphic differential (this follows from the Leibniz rule). This meromorphic differential is denoted  $df$ .

Let  $X$  be a Riemann surface.

### Definition 4.10

Let  $\omega$  be a meromorphic differential on  $X$ . The divisor  $\operatorname{div}(\omega)$  of  $\omega$  is defined as follows. If  $x \in X$  and  $\phi: U \rightarrow V$  is a coordinate chart around  $x$ , then  $(\operatorname{div}(\omega))(x)$  is the order of vanishing (negative if a pole) of  $\omega_V$  at  $\phi(x)$ .

Thus we see that if  $X$  is compact and connected, then  $\deg(\operatorname{div}(\omega))$  has the same value for any non zero meromorphic differential  $\omega$  (since the degree of the divisor of a meromorphic function vanishes).

A divisor of a non zero meromorphic differential on a compact and connected Riemann surface is called a *canonical divisor*.

- Complements.** (a) A non-singular plane algebraic curve over  $\mathbb{C}$  is connected when endowed with its canonical Riemann surface structure (the proof requires techniques outside the scope of this course).
- (b) A Riemann surface is an orientable surface when viewed as a real differentiable manifold of dimension 2. This follows from the fact the determinant of the Jacobian of a holomorphic function (viewed as a function of its real and imaginary part separately) is positive. This positivity follows from the Cauchy-Riemann equations (exercise).
- (c) By (b), a compact and connected Riemann surface is homeomorphic to a sphere with  $g$  handles (or  $g$ -holed torus) for some  $g \geq 0$ . The natural number  $g$  is called the *genus* of  $X$  and is written  $g(X)$ .
- (d) One can show the degree of a canonical divisor on a compact and connected Riemann surface  $X$  is  $2g(X) - 2$ . The proof involves triangulations and will not be given here.
- (e) There exists a non zero meromorphic function on any compact and connected Riemann surface (the proof requires techniques outside the scope of this course). We will not use this fact.

## Theorem 4.11 (Riemann-Hurwitz formula)

Let  $f : X \rightarrow Y$  be a non constant holomorphic map between compact and connected Riemann surfaces. Suppose that there exists a non zero meromorphic function on  $Y$ . Then we have

$$2g(X) - 2 = \deg(f)(2g(Y) - 2) + \sum_{p \in X} (\text{RD}(f, p) - 1).$$

**Proof.** Let  $h : Y \rightarrow \mathbb{C} \cup \{\infty\}$  be a non zero meromorphic function on  $Y$ . Let  $g := h \circ f$ . We shall compute  $\deg(\text{div}(dg)) = 2g(X) - 2$  in terms of  $h$  and  $f$ . For this, we need to compute  $\text{div}(dg)(p)$  at a point  $p \in X$  in terms of  $h$  and the local structure of  $f$  around  $p$ .

So let  $q := f(p)$ . Let  $\phi : U \rightarrow V$  be a coordinate chart of  $Y$  around  $q$  such that  $\phi(q) = 0$  and let  $\psi : U_1 \rightarrow V_1$  be a coordinate chart of  $X$  around  $p$  such that  $\psi(p) = 0$ . Suppose that  $(\phi \circ f \circ \psi^{-1})(w) = w^n$ , where  $n = n(p) = \text{RD}(f, p)$ .

We compute

$$\begin{aligned}\frac{d}{dw}(g \circ \psi^{-1}) &= \frac{d}{dw}(h \circ f \circ \psi^{-1}) = \frac{d}{dw}((h \circ \phi^{-1}) \circ \phi \circ f \circ \psi^{-1}) \\ &= \frac{d}{dw}((h \circ \phi^{-1})(w^n)) = nw^{n-1} \left( \frac{d}{dw}(h \circ \phi^{-1}) \right)(w^n) \quad (*)\end{aligned}$$

Now if

$$\left( \frac{d}{dw}(h \circ \phi^{-1}) \right)(w) = \sum_{k=k_0}^{\infty} a_k w^k$$

where  $a_{k_0} \neq 0$ , then  $k_0 = \text{div}(d(h))(q)$  and

$$(*) = n \sum_{k=k_0}^{\infty} a_k w^{nk+n-1}$$

so that

$$\text{div}(d(g))(p) = n \text{div}(d(h))(q) + n - 1 = \text{RD}(f, p) \text{div}(d(h))(q) + \text{RD}(f, p) - 1$$

We can thus compute

$$\begin{aligned} \deg(\operatorname{div}(g)) &= \sum_{p \in X} \operatorname{div}(d(g))(p) = \sum_{q \in Y} \sum_{p \in f^{-1}(\{q\})} \operatorname{div}(d(g))(p) \\ &= \sum_{q \in Y} \sum_{p \in f^{-1}(\{q\})} \left( \operatorname{RD}(f, p) \operatorname{div}(d(h))(q) + \operatorname{RD}(f, p) - 1 \right) \\ &= \sum_{q \in Y} \left( \left[ \sum_{p \in f^{-1}(\{q\})} \operatorname{RD}(f, p) \right] \cdot \operatorname{div}(d(h))(q) + \sum_{p \in f^{-1}(\{q\})} (\operatorname{RD}(f, p) - 1) \right) \\ &= \sum_{q \in Y} \left( \deg(f) \cdot \operatorname{div}(d(h))(q) + \sum_{p \in f^{-1}(\{q\})} (\operatorname{RD}(f, p) - 1) \right) \\ &= \deg(f) \deg(\operatorname{div}(d(h))) + \sum_{p \in X} (\operatorname{RD}(f, p) - 1) \end{aligned}$$

which is the required identity.  $\square$

We shall use the Riemann-Hurwitz formula in the proof of the following

### Theorem 4.12

Let  $C \subseteq \mathbb{P}^2(\mathbb{C})$  be a non singular curve of degree  $n$ . Then

$$g(C) = \frac{(n-1)(n-2)}{2}$$

**Proof.** Suppose wlog that  $[0, 0, 1] \notin C$ . Call  $\phi : C \rightarrow \mathbb{P}^1(\mathbb{C})$  the map sending  $[X, Y, Z]$  to  $[X, Y]$ .

The Riemann-Hurwitz formula for  $\phi$  states that

$$2g(C) - 2 = -2 \deg(\phi) + \sum_{p \in C} (\text{RD}(\phi, p) - 1).$$

We want to compute the left hand side from the right hand side, so we first need compute the ramification of the map  $\phi$  at a point  $p = [a, b, c] \in C$ .

Suppose wlog that  $a = 1$ , the calculation for  $b = 1$  being similar (note that by assumption  $(a, b) \neq (0, 0)$ ).

Let

$$w \mapsto [1, g(w), h(w)]$$

be a local parameter at  $p$ , with  $[1, g(0), h(0)] = p$ . We have the Euler relation

$$\begin{aligned} & P_x(1, g(w), h(w)) + g(w)P_y(1, g(w), h(w)) + h(w)P_z(1, g(w), h(w)) \\ &= nP(1, g(w), h(w)) = 0. \quad (*) \end{aligned}$$

and taking the derivative of the equation  $P(1, g(w), h(w)) = 0$ , we get

$$g'(w)P_y(1, g(w), h(w)) + h'(w)P_z(1, g(w), h(w)) = 0 \quad (**)$$

On the other hand, we have  $\phi([1, g(w), h(w)]) = [1, g(w)]$ , so the point  $p$  is ramified iff  $g'(0) = 0$ .

Now if  $g'(0) = 0$ , then  $h'(0) \neq 0$  since  $w \mapsto [1, g(w), h(w)]$  is a local parameter, so we conclude from (\*\*) that  $P_z(p) = 0$ .

On the other hand, if  $P_z(p) = 0$ , then from (\*\*) we see that either  $g'(0) = 0$  or that  $P_y(p) = 0$ .

In the latter case, we see from (\*) that  $P_x(p) = 0$ , which is not possible since  $C$  is non singular.

So we must have  $g'(0) = 0$ . So  $p$  is a ramification point of  $\phi$  iff  $P_z(p) = 0$ .

Now suppose that we have  $g'(0) = g''(0) = 0$ .

Taking the derivative of (\*\*), we get

$$\begin{aligned} & g''(w)P_y(1, g(w), h(w)) + g'(w)^2 P_{yy}(1, g(w), h(w)) \\ + & g'(w)h'(w)P_{yz}(1, g(w), h(w)) + h''(w)P_z(1, g(w), h(w)) \\ + & h'(w)g'(w)P_{zy}(1, g(w), h(w)) \\ + & h'(w)^2 P_{zz}(1, g(w), h(w)) = 0 \end{aligned}$$

and evaluating at 0, we get  $h'(0)^2 P_{zz}(a, b, c) = 0$  and hence  $P_{zz}(a, b, c) = 0$  since  $h'(0) \neq 0$ . Now let  $H = H_{x,y,z}$  be the Hessian matrix

$$\begin{bmatrix} P_{xx} & P_{xy} & P_{xz} \\ P_{yx} & P_{yy} & P_{yz} \\ P_{zx} & P_{zy} & P_{zz} \end{bmatrix}$$

We have the equation

$$H_{a,b,c}(a, b, c)^t = (n-1)(P_x(a, b, c), P_y(a, b, c), P_z(a, b, c))^t \quad (***)$$

(which corresponds to the Euler relations for  $P_x$ ,  $P_y$  and  $P_z$ ).

Let  $\underline{v}$  be an element of  $\mathbb{C}^3$  such that  $[\underline{v}]$  does not lie on the tangent line to  $C$  at  $p$ . Since  $p$  and  $[0, 0, 1]$  lie on the tangent line at  $p$  and are linearly independent, the three vectors

$$(a, b, c), (0, 0, 1), \underline{v}$$

form a basis of  $\mathbb{C}^3$ . As a bilinear form,  $H_{a,b,c}$  has the form

$$\begin{bmatrix} 0 & 0 & * \\ 0 & P_{zz} & \bullet \\ * & \bullet & \bullet \end{bmatrix}$$

in that basis (use  $(***)$  and the fact that  $P_z(a, b, c) = 0$ ). Here the entries marked  $*$  are no zero, and hence the determinant of this matrix is equal to (non zero number)  $\cdot P_{zz}(p)$ .

**We conclude that  $p = [a, b, c] \in C$  is a ramification point of  $\phi$  iff  $P_z(p) = 0$ , and is a point of ramification degree  $> 2$  iff  $P_z(p) = 0$  and  $p$  is an inflection point (ie one where  $\det(H) = 0$ , or equivalently one where  $P_{zz} = 0$ ).**

Now we go back to the problem of computing the genus of  $C$ .

First apply a projective transformation to  $C$  to ensure that the tangent lines at all its inflection points do not meet the point  $[0, 0, 1]$ , while keeping the condition that  $[0, 0, 1] \notin C$ . We these conditions in place, **we deduce from the above that  $\phi$  only has ramification points of degree 2.**

Since the Riemann-Hurwitz formula involves a sum over all the ramification points, we need to compute the number of ramification points. We shall apply Bézout's formula to do this.

In our situation, Bézout's formula says in particular that

$$\sum_{p \in C \cap Z(P_z)} I_p(C, Z(P_z)) = n(n-1) \quad (***)$$

So we need to compute  $I_p(C, Z(P_z))$  for each  $p \in C \cap Z(P_z)$ .

So let  $p = [a, b, c] \in C \cap Z(P_z)$ .

From Proposition 18 in the notes, we know that to check that  $I_p(C, Z(P_z)) = 1$ , we have to show that  $p$  is non singular in  $C$  and  $Z(P_z)$  and that  $T_p(C) \neq T_p(Z(P_z))$ .

We know that  $P_{zz}(p) \neq 0$ , so  $p$  is not singular in  $Z(P_z)$ . So we need to determine whether the vectors

$$(P_x(p), P_y(p), P_z(p))$$

and

$$(P_{zx}(p), P_{zy}(p), P_{zz}(p))$$

can be proportional.

Suppose that they are. We then have

$$H_{a,b,c}(0, 0, 1)^t = \lambda H_{a,b,c}(a, b, c)^t$$

for some  $\lambda \in \mathbb{C}^*$  by  $(***)$ , and thus  $a = b = 0$  since  $H_{a,b,c}$  is invertible by construction.

This contradicts the assumptions, so  $I_p(C, Z(P_z)) = 1$ .

**So we conclude from (\*\*\*) that  $\#(C \cap Z(P_z)) = n(n-1)$ , ie the number of ramification points of  $\phi$  is  $n(n-1)$ .**

Finally, since the Riemann-Hurwitz formula involves the degree of a map between Riemann surfaces, we need to compute the degree of  $\phi$ .

For this, consider the following. Suppose that

$$\phi([a, b, c]) = [A, B] \in \mathbb{P}^1(\mathbb{C}).$$

Then we have  $[a, b, c] = [A, B, c']$  for some  $c' \in \mathbb{C}$ .

So the elements of  $\phi^{-1}([A, B])$  correspond the roots in  $z$  of the equation  $P(A, B, z) = 0$ .

The polynomial  $P(A, B, z)$  is of degree  $n$  because  $\deg_z(P) = \deg(P)$ .

Furthermore, it has multiple roots iff there is  $\rho \in \mathbb{C}$  such that

$$P(A, B, \rho) = P_z(A, B, \rho) = 0.$$

So if we choose  $[A, B]$  so that  $\phi^{-1}([A, B])$  does not contain any ramification point of  $\phi$ , then  $\#\phi^{-1}([A, B]) = n$  and so by **Prop. 4.7**, we have  $\deg(\phi) = n$ .

We are now in a position to apply the Riemann-Hurwitz formula to  $\phi$ .

In our situation, it says that

$$2g(C) - 2 = n(-2) + n(n - 1)$$

or in other words that  $g(C) = (n - 1)(n - 2)/2$ .

This is the **genus-degree** formula.  $\square$

# The Riemann-Roch theorem

Let  $X$  be a compact and connected Riemann surface. We know that any meromorphic function with no poles on  $X$  is constant.

Indeed, such a function must reach a maximum on  $X$  because  $X$  is compact, and hence is constant in some open subset of  $X$  by the maximum principle, and hence on all of  $X$  by the identity principle.

The Riemann-Roch theorem improves on this result, by computing the dimension of the space of meromorphic functions with prescribed pole and zero behaviour at some points.

Specifically, let  $D$  be a divisor on  $X$ . We shall write  $\mathcal{L}(D)$  for the  $\mathbb{C}$ -vector space

$$\{f \text{ meromorphic on } X \mid (f) + D \geq 0\} \cup \{0\}$$

where we write  $D' \geq 0$  when a divisor  $D'$  has no negative values.

From the above, we have in particular that  $\mathcal{L}(0) \simeq \mathbb{C}$ , where  $0$  is the divisor on  $X$  with constant value  $0$ .

Let  $K$  be a canonical divisor on  $X$ . Let  $D$  be any divisor on  $X$ .

### Theorem 4.13 (Riemann-Roch)

- (a) *The space  $\mathcal{L}(D)$  is finite-dimensional as a  $\mathbb{C}$ -vector space.*
- (b) *We have*

$$\dim_{\mathbb{C}}(\mathcal{L}(D)) - \dim_{\mathbb{C}}(\mathcal{L}(K - D)) = \deg(D) + 1 - g(X)$$

One usually writes  $\ell(D)$  for  $\dim_{\mathbb{C}}(\mathcal{L}(D))$ .

In the notes, the Riemann-Roch theorem is proven in the situation where  $X$  is a plane algebraic curve. The proof is not examinable.

**Proof of (a).** Let  $S := \{x \in X \mid D(x) > 0\}$ .

If  $S = \emptyset$  then either  $\mathcal{L}(D) = 0$  or  $\mathcal{L}(D) \simeq \mathbb{C}$ , so we may assume that  $S$  is not empty.

Choose a local chart  $\phi_s : U_s \rightarrow V_s$  such that  $\phi_s(s) = 0$  for each  $s \in S$ .

For any  $s \in S$  and  $f$  meromorphic, write

$$(f \circ \phi_s^{-1})(w) = \sum_{k \geq -D(s)} a_k(f, s) w^k$$

Now define a linear map  $L : \mathcal{L}(D) \rightarrow \bigoplus_s \mathbb{C}^{D(x)}$  by the formula

$$L(f) := \bigoplus_s (a_{-D(s)}(f, s), a_{-D(s)+1}(f, s), \dots, a_{-1}(f, s)).$$

The kernel of  $L$  is at most one-dimensional since it consists of meromorphic functions with no poles. Hence  $\mathcal{L}(D)$  is finite dimensional.  $\square$

If  $f$  is a non zero meromorphic function on  $X$ , then  $\ell(D) = \ell(D + (f))$  (why?).

Also, if  $\deg(D) < 0$  then  $\mathcal{L}(D) = 0$ .

Indeed, if  $\deg(D) < 0$  and  $f$  is a non zero meromorphic function, then  $\deg((f) + D) = \deg(D) < 0$ , and so we cannot have  $(f) + D \geq 0$ , which would imply that  $\deg((f) + D) \geq 0$ .

Hence the Riemann-Roch theorem implies

#### Corollary 4.14

*Suppose that  $\deg(D) > \deg(K)$ . Then*

$$\dim_{\mathbb{C}}(\mathcal{L}(D)) = \deg(D) + 1 - g(X)$$

In particular, the RR theorem implies that there are non constant meromorphic functions on  $X$  (take  $D$  so that

$$\deg(D) > \max(2g(X) - 2, g(X)).$$

Applying the RR theorem to  $D = K$ , we get

$$\ell(K) - 1 = \deg(K) + 1 - g(X)$$

and so

$$\ell(K) = 1 + 2g(X) - 2 + 1 - g(X) = g(X)$$

To see the meaning of this, choose a non constant meromorphic function  $h$  on  $X$ . Then

$$\begin{aligned} \mathcal{L}(K) &\simeq \{f \text{ meromorphic} \mid (f) + \operatorname{div}(dh) \geq 0\} \\ &= \{f \text{ meromorphic} \mid \operatorname{div}(f \cdot dh) \geq 0\} \\ &= (\text{space of holomorphic differentials on } X) \end{aligned}$$

So on a curve a genus  $g$ , there are  $g$  linearly independent holomorphic differentials.

Finally, suppose that  $g(X) = 0$ .

Let  $p_0 \in X$ . Let  $D$  be the divisor such that  $D(p_0) = 1$  and such that  $D(p) = 0$  if  $p \neq p_0$ .

The RR theorem then gives

$$\ell(D) - \ell(K - D) = \deg(D) + 1 - g(X) = 2$$

and since

$$\deg(K - D) = 2g(X) - 2 - 1 = -3,$$

we have  $\ell(K - D) = 0$ .

Thus we have  $\ell(D) = 2$ , so  $\mathcal{L}(D)$  does not only consist of constant functions.

Let  $f \in \mathcal{L}(D)$  be a non constant function. Then we have from the definition

$$(\text{divisor of zeroes of } f) \geq (\text{pole divisor of } f) - D.$$

In particular,  $f$  has a pole of order 1 at  $D$  and no other poles.

In other words, the fibre at  $\infty$  of  $f$  consists of one unramified point.

This implies that  $f$  is of degree one, and in particular that  $f$  has no ramification points (by Prop. 4.7).

This in turn implies that  $f^{-1}$  is also holomorphic, and so we have  $X \simeq \mathbb{P}^1(\mathbb{C})$ .

So any Riemann surface of genus 0 is isomorphic to the projective line.