A12162H1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part B: Paper B5.2
Honour School of Mathematics and Statistics Part B: Paper B5.2

APPLIED PARTIAL DIFFERENTIAL EQUATIONS

TRINITY TERM 2025

Tuesday 17 June, 2:30pm to 4:15pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. (a) [15 marks] Consider solutions $h(x,t)$ of the partial differential equation

$$h_t - (h^n h_{xxxxx})_x = 0 \qquad \text{on } -s(t) < x < s(t) \tag{1.1}$$

that satisfy the following boundary conditions at $x = \pm s(t)$:

$$h(\pm s(t), t) = 0, \tag{1.2}$$
$$h_x(\pm s(t), t) = 0, \tag{1.3}$$
$$h_{xx}(\pm s(t), t) = 0, \tag{1.4}$$
$$h^n h_{xxxxx} \to 0 \qquad \text{for } x \to \pm s(t), \tag{1.5}$$

for $t > 0$, where $n > 0$ is a constant.

(i) Show that solutions $h(x,t)$ and $s(t)$ of (1.1)-(1.5) conserve

$$I = \int_{-s(t)}^{s(t)} h(x,t)\,\mathrm{d}x.$$

(ii) Show that for a suitable choice of a, b and c, the system of equations (1.1)-(1.5) is invariant under the scalings

$$t = \varepsilon^a \bar{t}, \quad x = \varepsilon^b \bar{x}, \quad s(t) = \varepsilon^b \bar{s}(\bar{t}), \quad h(x,t) = \varepsilon^c \bar{h}(\bar{x}, \bar{t}).$$

for all $\varepsilon > 0$. Use this result to determine constants α and β so that

$$h(x,t) = t^\alpha \phi(\xi) \quad \text{with } \xi = x/t^\beta \text{ and } s(t) = \sigma t^\beta$$

is a self-similar solution of (1.1)-(1.5) for which $I = 1$. In particular, show that $\alpha = -\beta$. State the resulting boundary value problem for a fifth order ordinary differential equation for ϕ and σ.

(iii) Determine the self-similar solution explicitly in the case $n = 1$.

(b) [10 marks] Consider the Neumann problem on the bounded domain $\Omega \subset \mathbb{R}^2$ for the two dimensional Laplace operator

$$\nabla^2 u = f \quad \text{on } \Omega, \quad u_n = g \quad \text{on } \partial\Omega. \tag{1.6}$$

Here, u_n denotes the derivative of u with respect to the outward normal $\boldsymbol{n}$ of $\partial\Omega$, and f and g satisfy the solvability condition

$$\int_\Omega f\,\mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} g\,\mathrm{d}s$$

for positively oriented $\partial\Omega$.

(i) State the boundary value problem for the *modified* Green's function $G(\boldsymbol{x}, \boldsymbol{\xi})$.

(ii) Now assume that $\Omega \subset \mathbb{R}^2$ is the *unit disk*. Obtain a solution for G using the method of images with the image singularity at $\boldsymbol{\xi'} = \boldsymbol{\xi}/|\boldsymbol{\xi}|^2$. You must show clearly that all conditions for the Green's function are satisfied.
[*The 2D free space Green's function is given by*

$$H(\boldsymbol{x}; \boldsymbol{\xi}) = \frac{1}{2\pi} \ln|\boldsymbol{x} - \boldsymbol{\xi}|.$$

You may use this expression without proof.]

2. Consider the first order quasilinear partial differential equation

$$u_t + \frac{1}{4}(u^4)_x = 0 \tag{2.1}$$

for $t > 0$.

(a) [13 marks] (i) State the characteristic equations for (2.1) with initial data corresponding to $u(x, 0) = u_0(x)$, $a < x < b$, for continuous u_0. Obtain the solution for this system in parametric form.

(ii) State the condition for the speed v of a shock for (2.1) in terms of its left and right state u_- and u_+, respectively. State the condition for the causality of the shock in terms of u_-, u_+ and v.

(iii) Determine the causal solution for (2.1) with initial data

$$u(x, 0) = \begin{cases} 0 & \text{if } x < 0, \\ -1 & \text{if } x > 0. \end{cases}$$

(iv) Determine the causal solution for (2.1) with initial data

$$u(x, 0) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases}$$

(b) [12 marks] Now consider the causal solution of (2.1) with initial data

$$u(x, 0) = \begin{cases} 0 & \text{if } x < 0, \\ -1 & \text{if } 0 < x < 1, \\ 0 & \text{if } 1 < x. \end{cases}$$

Determine the explicit solution for $u(x, t)$ for $t < t_1 = 4/3$. Why does this form of the solution cease to be valid at $t = t_1$? Continue the solution to $t > t_1$, giving the explicit form for $u(x, t)$.

3. Consider the nonlinear partial differential equation

$$u_x + y\,(u_y)^2 - u = 0, \tag{3.1}$$

together with the initial data $u(x,1) = 1$ for $x > 0$.

(a) [6 marks] Formulate Charpit's equations for (3.1). State the associated initial conditions, distinguishing carefully between the cases for which (A) $u_y(x,1) > 0$ and (B) $u_y(x,1) < 0$.

(b) [6 marks] Obtain the parametric solution of (3.1) with $u_y(x,1) > 0$ for $x > 0$ (i.e., case (A)). State and sketch the region in which the solution $u(x,y)$ exists and is uniquely determined by the initial data. Determine $u(x,y)$.

(c) [8 marks] Obtain the parametric solution of (3.1) with $u_y(x,1) < 0$ for $x > 0$ (i.e., case (B)). State and sketch the region in which the solution $u(x,y)$ exists and is uniquely determined by the initial data. Determine $u(x,y)$.

(d) [5 marks] If the initial data is instead $u(x,1) = \mathrm{e}^x$ for $x > 0$, explain clearly how the solution structure is changed. Comment on both the number of solutions and where they are uniquely defined by the initial data.

[*You may wish to note that* $u(x,y) = \left(\sqrt{y} - 1\right)^2$ *satisfies the equation* (3.1).]