A12165H1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part B: Paper B5.4
Honour School of Mathematics and Statistics Part B: Paper B5.4

WAVES AND COMPRESSIBLE FLOW

TRINITY TERM 2025

Thursday 5 June, 9:30am to 11:15am

You may submit answers to as many questions as you wish but only the best two will count for the total mark. All questions are worth 25 marks.

You should ensure that you observe the following points:

- start a new answer booklet for each question which you attempt.
- indicate on the front page of the answer booklet which question you have attempted in that booklet.
- cross out all rough working and any working you do not want to be marked. If you have used separate answer booklets for rough work please cross through the front of each such booklet and attach these answer booklets at the back of your work.
- hand in your answers in numerical order.

If you do not attempt any questions, you should still hand in an answer booklet with the front sheet completed.

Do not turn this page until you are told that you may do so

1. Heat is supplied at a rate $q(\boldsymbol{x},t)$ per unit mass to an inviscid gas with constant specific heat capacity $c_v > 0$, thermal conductivity $k > 0$, and ratio of specific heats $\gamma > 1$. In the absence of any body force, the density $\rho(\boldsymbol{x},t)$, pressure $p(\boldsymbol{x},t)$, temperature $T(\boldsymbol{x},t)$, and velocity $\boldsymbol{u}(\boldsymbol{x},t)$ satisfy the *Euler equations*, the *energy equation* and the *ideal gas law*

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \qquad \rho c_v\left(\frac{\partial T}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}T\right) = -p\boldsymbol{\nabla}\cdot\boldsymbol{u} + k\nabla^2 T + \rho q,$$

$$\rho\left(\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\right) = -\boldsymbol{\nabla}p, \qquad p = (\gamma-1)\rho c_v T.$$

(a) [5 marks] Define the *entropy* S per unit mass and show that it satisfies the equation

$$\rho T\left(\frac{\partial S}{\partial t} + \boldsymbol{u}\cdot\boldsymbol{\nabla}S\right) = k\nabla^2 T + \rho q.$$

(b) [14 marks] The gas is contained in a two-dimensional box $(0,a)\times(0,b)$, whose walls are rigid, impermeable, and thermally insulated. The gas is initially at rest with $q = 0$ and constant pressure p_0, density ρ_0 and temperature T_0. The gas is disturbed by the application of a small heating source, such that $q = q_1$, $p = p_0 + p_1$, $\rho = \rho_0 + \rho_1$, $T = T_0 + T_1$ and $\boldsymbol{u} = \boldsymbol{u}_1$, where nonlinear terms involving the small variables with subscript 1 are negligible.

Show that the flow is irrotational, and that the pressure and temperature perturbations satisfy

$$p_1 = -\rho_0\frac{\partial \phi}{\partial t}, \qquad (\gamma-1)\rho_0 c_v T_1 = -\rho_0\frac{\partial \phi}{\partial t} - \frac{p_0}{\rho_0}\rho_1,$$

where ϕ is the velocity potential. Hence show that ϕ satisfies the equation

$$\kappa\nabla^2\left(\gamma\frac{\partial^2\phi}{\partial t^2} - c_0^2\nabla^2\phi\right) - \gamma\frac{\partial}{\partial t}\left(\frac{\partial^2\phi}{\partial t^2} - c_0^2\nabla^2\phi\right) = \gamma(\gamma-1)\frac{\partial q}{\partial t},$$

where $c_0^2 = \gamma p_0/\rho_0$ and $\kappa = k/\rho_0 c_v$.

Derive the boundary conditions

$$\frac{\partial\phi}{\partial x} = \frac{\partial^3\phi}{\partial x^3} = 0 \quad \text{at } x = 0,\, a,$$

and the corresponding conditions at $y = 0,\, b$. [*Hint: the temperature perturbation satisfies* $\boldsymbol{n}\cdot\boldsymbol{\nabla}T_1 = 0$ *at each insulated boundary, where* $\boldsymbol{n}$ *is a unit normal.*]

(c) [6 marks] (i) Consider the case where $q(x,y,t) = \mathrm{Re}\left[A\cos(\pi x/a)\mathrm{e}^{-\mathrm{i}\omega t}\right]$, with $A \in \mathbb{C}$ and $\omega > 0$, where $\mathrm{Re}[\cdot]$ denotes the real part. Show that time-periodic solutions with $\phi(x,y,t) = \mathrm{Re}\left[B\cos(\pi x/a)\mathrm{e}^{-\mathrm{i}\omega t}\right]$ are possible, and solve for the response amplitude B.

(ii) To examine the limit of weak thermal diffusivity, let $\kappa = \epsilon a c_0/\pi$, where $0 < \epsilon \ll 1$, and suppose that ω is close to a resonant frequency, so $\omega = \pi c_0/a(1+\epsilon\omega_1)$. Show that the response amplitude is given approximately by

$$|B| \sim \frac{\gamma(\gamma-1)a^2|A|}{\epsilon\pi^2 c_0^2\sqrt{4\gamma^2\omega_1^2 + (\gamma-1)^2}}.$$

2. In an inviscid barotropic fluid, the pressure p and density ρ are related by $p = P(\rho)$, where P is a specified monotonic increasing function. The fluid undergoes steady two-dimensional irrotational flow, with velocity given by $\boldsymbol{u} = \boldsymbol{\nabla}\phi$, where $\phi(x, y)$ is the velocity potential. There is negligible body force.

The fluid flows parallel to a wall along the x-axis, with $\phi \to Ux$ and $\rho \to \rho_0$ in the far field, where U and ρ_0 are positive constants. The wall is flat except for a small bump close to $x = 0$, defined by

$$y = F(x) = \begin{cases} 0 & |x| \geqslant a, \\ f(x) & |x| < a, \end{cases}$$

where f/a is small and $f(\pm a) = f'(\pm a) = 0$.

(a) [7 marks] Define the *speed of sound* $c(\rho)$, and show that c and ϕ satisfy the equations

$$\frac{c^2}{\rho}\boldsymbol{\nabla}\rho + \boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{\nabla}\phi|^2\right) = \boldsymbol{0},$$

$$\left(\frac{\partial\phi}{\partial x}\right)^2\frac{\partial^2\phi}{\partial x^2} + 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} + \left(\frac{\partial\phi}{\partial y}\right)^2\frac{\partial^2\phi}{\partial y^2} = c^2\nabla^2\phi.$$

[*You may use without proof the identity* $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{\nabla}\left(\frac{1}{2}|\boldsymbol{u}|^2\right) + (\boldsymbol{\nabla}\times\boldsymbol{u})\times\boldsymbol{u}$.]

State the boundary condition on the wall.

(b) [8 marks] Consider small perturbations about the far-field flow due to the bump by setting $\phi = Ux + \phi_1$, $\rho = \rho_0 + \rho_1$, and linearising with respect to $\{\phi_1, \rho_1, f\}$. Show that ϕ_1 satisfies

$$\left(1 - M^2\right)\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0 \qquad y > 0,$$

$$\frac{\partial\phi_1}{\partial y} = UF'(x) \qquad y = 0,$$

where $M = U/c_0$ and c_0 is the speed of sound in the undisturbed gas. Show also that the lowest-order drag (i.e., the force in the x-direction) acting on the bump is given by

$$D = -\rho_0 U \int_{-a}^{a} f'(x)\frac{\partial\phi}{\partial x}(x, 0)\,\mathrm{d}x.$$

(c) [10 marks] Now assume that the background flow is subsonic and impose the far-field condition $\boldsymbol{\nabla}\phi_1(x, y) \to \boldsymbol{0}$ as $x^2 + y^2 \to \infty$.

(i) Show that

$$\phi_1(x, y) = \frac{U}{2\pi\beta}\int_{-a}^{a} f'(s)\log\left((x - s)^2 + \beta^2 y^2\right)\mathrm{d}s,$$

where $\beta = \sqrt{1 - M^2}$.

[*Standard properties of the Fourier transform may be used without proof, as well as the result that* $\pi\mathrm{e}^{-y|k|}$ *is the Fourier transform of* $y/\left(x^2 + y^2\right)$ *when* $y > 0$. *You may find it helpful to solve for* $\partial\phi_1/\partial y$ *before evaluating* ϕ_1.]

(ii) Hence show that $D = 0$ whenever the bump is symmetric, i.e., whenever f is an even function.

3. A shallow layer of inviscid, incompressible fluid flows steadily over an impermeable horizontal substrate, subject to a transverse gravitational acceleration g. The fluid layer height $h(x,y) > 0$ and velocity $\big(u(x,y), v(x,y)\big)^{\mathrm{T}}$ satisfy the following weak formulation of the shallow-water equations:

$$\oint_C \mathbf{P}\,\mathrm{d}y - \mathbf{Q}\,\mathrm{d}x = \mathbf{0} \quad \text{for all piecewise smooth simple closed curves } C,$$

where

$$\mathbf{P} = \begin{pmatrix} hu \\ hu^2 + gh^2/2 \\ huv \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} hv \\ huv \\ hv^2 + gh^2/2 \end{pmatrix}.$$

(a) [8 marks] (i) Show that

$$\frac{\partial \mathbf{P}}{\partial x} + \frac{\partial \mathbf{Q}}{\partial y} = \mathbf{0}$$

in any simply-connected open region of the (x,y)-plane in which $\{h,u,v\}$ are continuously differentiable.

(ii) Derive the *Rankine–Hugoniot relations*

$$\big[\mathbf{Q}\big]_-^+ \cos\beta = \big[\mathbf{P}\big]_-^+ \sin\beta$$

satisfied across a shock making an angle β with the x-axis, where $[\cdot]_-^+$ denotes the jump in $\cdot$ from one side of the shock to the other.

(b) [8 marks] Consider fluid crossing a plane shock parallel to the y-axis, with $(h,u,v) = (h_-, u_-, v_-)$ in $x < 0$ and $(h,u,v) = (h_+, u_+, v_+)$ in $x > 0$. Assume that $\{h_-, u_-, v_-\}$ are all specified and positive, while $\{h_+, u_+, v_+\}$ are unknown *a priori*.

Explain briefly why the height h of the fluid layer must increase as the flow crosses the shock.

Show that $v_+ = v_-$ and

$$\frac{h_+}{h_-} = \frac{1}{2}\left(\sqrt{1 + 8F_-^2} - 1\right),$$

where $F_- = u_-/\sqrt{gh_-} > 0$. Deduce the inequalities $u_-/u_+ > 1$ and $F_- > 1$.

(c) [9 marks] Show that the flow is deflected by an angle δ as it passes through the shock, where

$$\tan\delta = \frac{(u_- - u_+)v_-}{u_- u_+ + v_-^2}.$$

Show that the deflection angle obeys the upper bound

$$\sin\delta \leqslant \frac{1 + 2F_-^2 - \sqrt{1 + 8F_-^2}}{2F_-^2}.$$