

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C6.3b

APPLIED COMPLEX VARIABLES

Trinity Term 2012

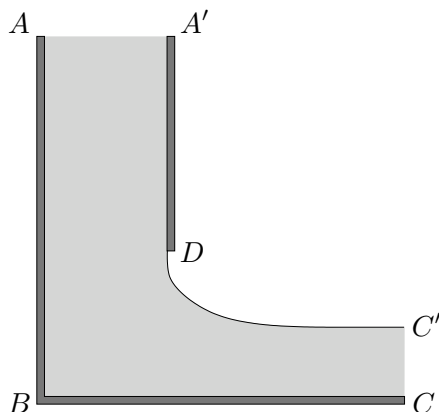
Thursday, 31 May 2012, 2.30pm to 4.00pm

You may submit answers to as many questions as you wish; the best two will count for the total mark.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

Do not turn this page until you are told that you may do so

1. Consider the steady two-dimensional potential flow illustrated below, in which fluid is draining through a hole in the side of a container.



The points A , B , C , C' , D and A' lie at $(0, \infty)$, $(0, 0)$, $(\infty, 0)$, (∞, a) , $(1, h)$ and $(1, \infty)$ respectively, where $0 < a < 1$ and $a < h$. The stream function ψ takes the value 0 on the fixed straight walls AB and BC , and it takes the value a on the fixed straight wall $A'D$ and on the free surface DC' . The free surface DC' is tangent at D to the wall $A'D$. As $x \rightarrow \infty$ with $0 < y < a$ the fluid velocity $(u, v) \rightarrow (1, 0)$, so that $u^2 + v^2 = 1$ on DC' . As $y \rightarrow \infty$ with $0 < x < 1$ the fluid velocity $(u, v) \rightarrow (0, -a)$. Take the potential ϕ to be zero at the point B .

- (a) Show that the flow domain in the potential plane ($w = \phi + i\psi$) is a strip, while that in the hodograph plane ($w' = u - iv$) is a quarter circle.
- (b) Map each of these domains to the upper half ζ -plane, taking AA' to $\zeta = 0$, B to $\zeta = 1$ and CC' to $\zeta = \infty$. Hence, show that

$$(1 - b^2)e^{\pi w/a} + b^2 = \left(\frac{1 + (w')^2}{1 - (w')^2} \right)^2,$$

where b is a constant that you should determine in terms of a .

- (c) Parametrize the free surface by $w' = e^{-i\theta}$. Show that, on the free surface,

$$\frac{dz}{d\theta} = -\frac{2ae^{i\theta} \cot \theta}{\pi(\cos^2 \theta + b^2 \sin^2 \theta)}.$$

- (d) Find the parametric equations for the free surface in terms of integrals, and hence show that

$$\frac{h}{a} = 1 + \frac{2}{\pi} \int_0^{\pi/2} \frac{\cos t}{\cos^2 t + b^2 \sin^2 t} dt.$$

What happens as $a \rightarrow 1$?

2. (a) Let Γ be a contour in the complex plane and

$$w(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Show that if f is continuous on Γ , t is any point at which Γ is smooth and f is holomorphic in a neighbourhood of t , then the limiting values of $w(z)$ as Γ is approached from either side are $w_{\pm}(t)$, where

$$w_{\pm}(t) = \pm \frac{1}{2} f(t) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - t} d\zeta$$

and you should define the integral \int precisely.

- (b) Let $\Gamma = \{x + iy : |x| < 1, y = 0\}$ and $\bar{\Gamma} = \{x + iy : |x| \leq 1, y = 0\}$. Suppose that $w(z)$ is holomorphic away from $\bar{\Gamma}$ and $\operatorname{Re}(w_{\pm}(x)) = g(x)$ on Γ for some known smooth real function $g(x)$. Suppose that $\tilde{w}(z)$ is holomorphic and non-zero away from $\bar{\Gamma}$ and $\tilde{w}_+(x) = -\tilde{w}_-(x) \neq 0$ on Γ . Determine the density $F(\xi)$ for which a solution for $w(z)$ is given by

$$\frac{w(z)}{\tilde{w}(z)} = \frac{1}{2\pi i} \int_{-1}^1 \frac{F(\xi)}{\xi - z} d\xi.$$

Deduce that

$$f(x) = \frac{2\tilde{w}_+(x)}{\pi i} \int_{-1}^1 \frac{g(\xi)}{\tilde{w}_+(\xi)(\xi - x)} d\xi$$

is a solution of the Cauchy singular integral equation

$$\frac{1}{2\pi i} \int_{-1}^1 \frac{f(\xi)}{\xi - x} d\xi = g(x) \quad \text{for } |x| < 1.$$

- (c) By defining an appropriate branch of

$$w(z) = \log \left(\frac{z-1}{z+1} \right),$$

show that

$$\int_{-1}^1 \frac{dt}{t-x} = \log \left(\frac{1-x}{1+x} \right) \quad \text{for } |x| < 1.$$

- (d) Evaluate

$$\int_{-1}^1 \log \left(\frac{1-t}{1+t} \right) \frac{dt}{t-x} \quad \text{for } |x| < 1.$$

3. (a) Suppose $w_+(z)$ is holomorphic and bounded at infinity in $\text{Im}(z) > \alpha$ and $w_-(z)$ is holomorphic and bounded at infinity in $\text{Im}(z) < \beta$.

- (i) Suppose $w_+(z) = w_-(z)$ in $\alpha < \text{Im}(z) < \beta$. Show that $w_+(z) = w_-(z)$ is a constant.
(ii) Suppose $w_+(z) - w_-(z) = G(z)$ in $\alpha < \text{Im}(z) < \beta$, where G is holomorphic in $\alpha < \text{Im}(z) < \beta$ and $G \rightarrow 0$ as $|z| \rightarrow \infty$ with $\alpha < \text{Im}(z) < \beta$. Show that $w_+(z) - G_+(z) = w_-(z) - G_-(z)$ is a constant, where

$$G_{\pm}(z) = \frac{1}{2\pi i} \int_{\Gamma_{\pm}} \frac{G(\zeta)}{\zeta - z} d\zeta$$

and Γ_{\pm} should be defined.

- (iii) Suppose $F(z)w_+(z) = w_-(z)$ in $\alpha < \text{Im}(z) < \beta$, where F is holomorphic and non-zero in $\alpha < \text{Im}(z) < \beta$ and $F \rightarrow 1$ as $|z| \rightarrow \infty$ with $\alpha < \text{Im}(z) < \beta$. Show that $F_+(z)w_+(z) = F_-(z)w_-(z)$ is a constant, where $F_{\pm}(z)$ should be defined.

(b) Suppose

$$\nabla^2 u = u \quad \text{in } y > 0,$$

with

$$u = 1 \quad \text{on } y = 0, x > 0, \quad \frac{\partial u}{\partial y} = 0 \quad \text{on } y = 0, x < 0,$$

and $u \rightarrow 0$ as $x^2 + y^2 \rightarrow \infty$. Define

$$f_-(x) = \begin{cases} u(x, 0) & \text{for } x < 0, \\ 0 & \text{for } x > 0, \end{cases} \quad g_+(x) = \begin{cases} 0 & \text{for } x < 0, \\ \frac{\partial u}{\partial y}(x, 0) & \text{for } x > 0. \end{cases}$$

Suppose $f_-(x) = O(e^{bx})$ as $x \rightarrow -\infty$ and $g_+(x) = O(e^{ax})$ as $x \rightarrow \infty$, where a and b are constants satisfying $a \leq 0$ and $b \geq 1$.

- (i) By taking a Fourier transform show that

$$\bar{g}_+(k) + (k^2 + 1)^{1/2} \left(\bar{f}_-(k) + \frac{i}{k} \right) = 0 \quad \text{for } 0 < \text{Im}(k) < 1, \quad (1)$$

where you should define precisely the branch of $(k^2 + 1)^{1/2}$.

- (ii) Deduce from (1) expressions for $\bar{f}_-(k)$ and $\bar{g}_+(k)$, defining precisely the branch of each multi-valued function that you use.
(iii) Sketch inversion contours Γ^{\pm} for which

$$f_-(x) = \frac{1}{2\pi} \int_{\Gamma^-} \bar{f}_-(k) e^{-ikx} dk, \quad g_+(x) = \frac{1}{2\pi} \int_{\Gamma^+} \bar{g}_+(k) e^{-ikx} dk.$$

[You may use the fact that $k^{1/2} \bar{g}_+(k)$ and $k \bar{f}_-(k)$ are bounded as $|k| \rightarrow \infty$ without proof.]

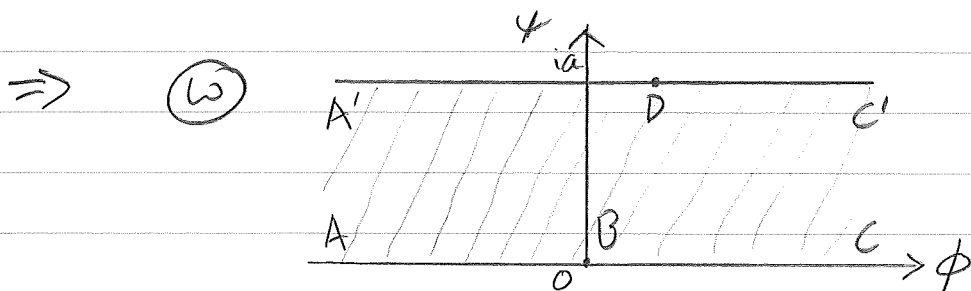
(6.3b)

(a) Potential plane bounded by $\psi = 0$ and $\psi = a$ (by maximum principle)

At AA' , $w' \rightarrow ia \Rightarrow w \sim ia z \Rightarrow \phi \sim -ay \rightarrow -\infty$ as $y \rightarrow \infty$.

At B , $\phi = 0$

At CC' , $w' \rightarrow 1 \Rightarrow w \sim z \Rightarrow \phi \sim x \rightarrow \infty$ as $x \rightarrow \infty$



AA' : $u = 0, v = -a$

AB : $u = 0, -a \leq v \leq 0$

B : $u = 0, v = 0$ (stagnation point)

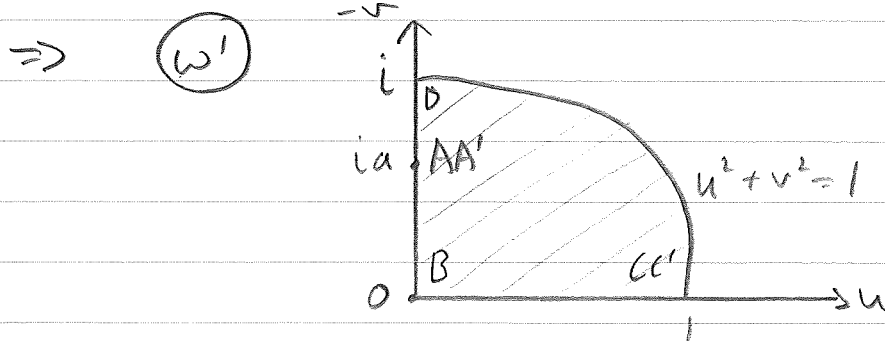
BC : $0 \leq u \leq 1, v = 0$

CC' : $u = 1, v = 0$

$C'D$: $u^2 + v^2 = 1, u \geq 0, v \leq 0$

D : $u = 0, v = -1$

DA' : $u = 0, -1 \leq v \leq -a$

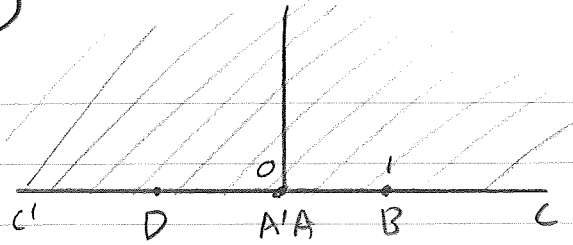


1(b)

(w)

$$z = e^{nw/a}$$

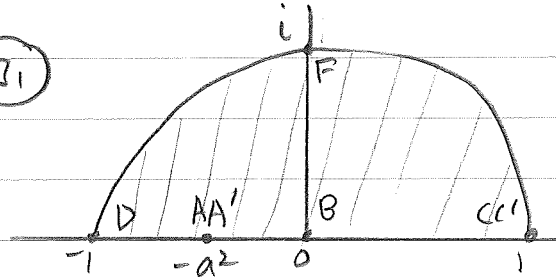
(z)



(w')

$$z_1 = (w')^2$$

(z₁)



(z₂)

$$z_2 = \frac{1+z_1}{1-z_1}$$

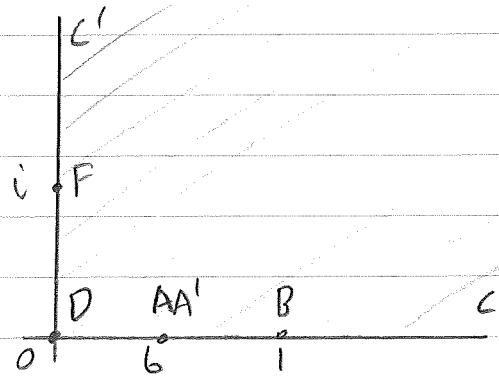
$$-a^2 \rightarrow b = \frac{1-a^2}{1+a^2} \in (0,1)$$

$$0 \rightarrow 1$$

$$1 \rightarrow \infty$$

$$i \rightarrow i$$

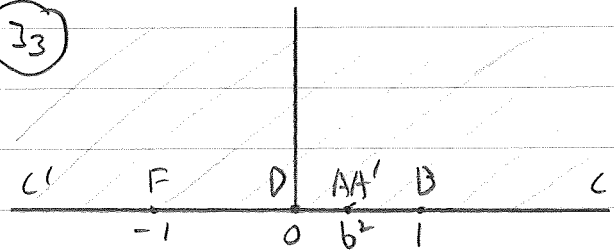
(z₂)



(z₃)

$$z_3 = z_2^2$$

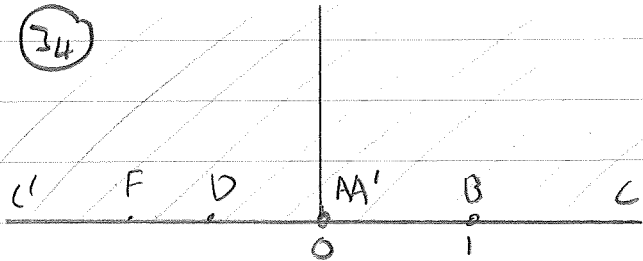
(z₃)



(z₄)

$$z_4 = \frac{z_3 - b^2}{1 - b^2}$$

(z₄)



Same points ABC in z- and z₄-planes $\Rightarrow z = z_4$

$$\Rightarrow (1-b^2)e^{nw/a} + b^2 = z_3 = \left(\frac{1 + (w')^2}{1 - (w')^2} \right)^2$$

$$1(c) \quad z = e^{\pi w/a} = \frac{1}{1-b^2} \left(\left(\frac{1+(w')^2}{1-(w')^2} \right)^2 - b^2 \right)$$

$$\frac{dz}{d\theta} = \frac{dz}{dw} \frac{dw}{dz} \frac{dz}{d\theta} = \frac{1}{w' z_w} z_\theta,$$

$$\text{where } w' = e^{-i\theta} \Rightarrow$$

$$z = \frac{1}{1-b^2} \left(\left(\frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}} \right)^2 - b^2 \right) = \frac{b^2 + \cot^2 \theta}{b^2 - 1}$$

$$\Rightarrow z_\theta = \frac{-2 \cot \theta \operatorname{cosec}^2 \theta}{b^2 - 1}$$

$$\text{Also, } z_w = \frac{\pi}{a} e^{\pi w/a} = \frac{\pi z}{a}.$$

Hence,

$$\frac{dz}{d\theta} = e^{i\theta} \frac{a}{\pi z} z_\theta$$

$$= e^{i\theta} \frac{a(b^2 - 1)}{\pi(b^2 + \cot^2 \theta)} \cdot \frac{-2 \cot \theta \operatorname{cosec}^2 \theta}{b^2 - 1}$$

$$= - \frac{2a e^{i\theta} \cot \theta}{\pi(\cos^2 \theta + b^2 \sin^2 \theta)}$$

1(d) Since $x=1, y=h, \theta = -\frac{\pi}{2}$ at D ,

$$x(\theta) = \underbrace{x(-\pi/2)}_{=1} - \frac{2a}{\pi} \int_{-\pi/2}^{\theta} \frac{\cos \hat{\theta} \cot \hat{\theta}}{\cos^2 \hat{\theta} + b^2 \sin^2 \hat{\theta}} d\hat{\theta}$$

$$y(\theta) = \underbrace{y(-\pi/2)}_{=h} - \frac{2a}{\pi} \int_{-\pi/2}^{\theta} \frac{\cos \hat{\theta}}{\cos^2 \hat{\theta} + b^2 \sin^2 \hat{\theta}} d\hat{\theta}$$

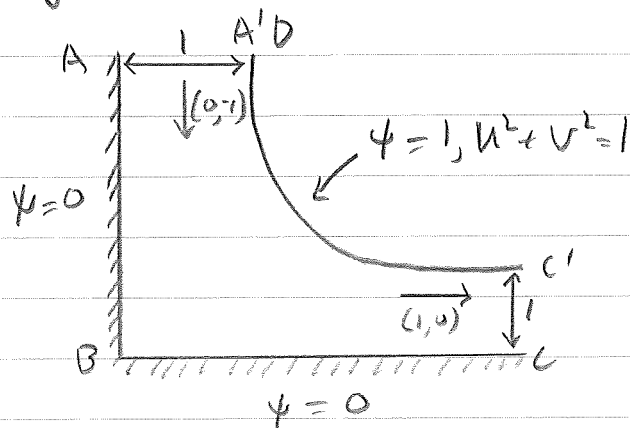
At C' , $y=a, \theta=0 \Rightarrow y(0)=a$

$$\Rightarrow a = h - \frac{2a}{\pi} \int_{-\pi/2}^0 \frac{\cos \hat{\theta}}{\cos^2 \hat{\theta} + b^2 \sin^2 \hat{\theta}} d\hat{\theta}$$

$$\Rightarrow h = a + \frac{2a}{\pi} \int_0^{\pi/2} \frac{\cos t}{\cos^2 t + b^2 \sin^2 t} dt$$

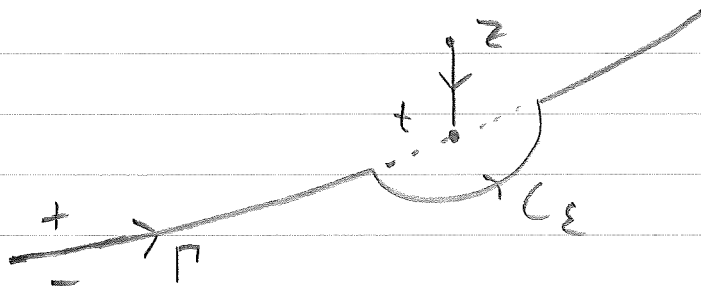
($\hat{\theta} = -t$)

As $a \rightarrow 1, b \rightarrow 0 \Rightarrow h \rightarrow \infty$, i.e. $D \rightarrow A'$, giving the flow



2(a) Label the LHS of Γ as "+" and the RHS as "-", oriented in the direction of integration.

As $z \rightarrow t \in \Gamma$ from the + side indent Γ with a small (approximate) semi-circle C_ε around t as shown, where the radius ε is sufficiently small that f is holomorphic in $D(t, 2\varepsilon) = \{z: |z-t| < 2\varepsilon\}$



Let $\gamma_\varepsilon = \Gamma \cap D(t, \varepsilon)$, i.e. the portion of Γ replaced by C_ε .

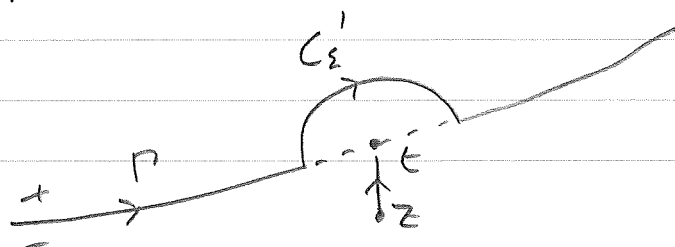
$$\text{Deformation thm} \Rightarrow w(z) = \frac{1}{2\pi i} \left(\int_{\Gamma \setminus \gamma_\varepsilon} + \int_{C_\varepsilon} \right) \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$\Rightarrow_{(z \rightarrow t)} w_+(t) = \frac{1}{2\pi i} \left(\int_{\Gamma \setminus \gamma_\varepsilon} + \int_{C_\varepsilon} \right) \frac{f(\zeta)}{\zeta - t} d\zeta$$

$$\Rightarrow_{(\varepsilon \rightarrow 0)} w_+(t) = \underbrace{\frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - t} d\zeta}_{\frac{f}{\Gamma} := \lim_{\varepsilon \rightarrow 0} \int_{\Gamma \setminus \gamma_\varepsilon}} + \underbrace{\frac{1}{2} \cdot \frac{2\pi i \operatorname{res}}{2\pi i} \frac{f(\zeta)}{\zeta - t}}_{= \frac{1}{2} f(t)}$$

The PVI exists \because log singularities cancel as $\varepsilon \rightarrow 0$ by cty.

For $w_-(t)$ replace γ_ε with C'_ε as shown:



Semi-circle now gives a contribution

$$-\frac{1}{2} \frac{2\pi i}{2\pi i} \operatorname{res}_{z=t} \frac{f(z)}{z-t} = -\frac{1}{2} f(t)$$

Hence,

$$w_{\pm}(t) = \pm \frac{1}{2} f(t) + \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z-t} dz \quad (\text{PF})$$

□

2(b) Seek a solution by setting

$$w(z) = \frac{1}{2\pi i} \int_{-1}^1 \frac{f(\xi)}{\xi-z} d\xi, \quad \frac{w(z)}{\tilde{w}(z)} = W(z) = \frac{1}{2\pi i} \int_{-1}^1 \frac{F(\xi)}{\xi-z} d\xi$$

where the densities f and F are TBD.

$$(\text{PF}) \Rightarrow w_{\pm} = \pm \frac{1}{2} f + h, \quad W_{\pm} = \pm \frac{1}{2} F + H \text{ on } \Gamma$$

$$\text{where } h(x) = \frac{1}{2\pi i} \int_{-1}^1 \frac{f(\xi)}{\xi-x} d\xi, \quad H(x) = \frac{1}{2\pi i} \int_{-1}^1 \frac{F(\xi)}{\xi-x} d\xi$$

for $|x| < 1$.

Now, $\operatorname{Re}(w_{\pm}) = g$ on $\Gamma \Rightarrow w_{\pm} = g + i v_{\pm}$ on Γ , v_{\pm} real.

$$\text{Subtract} \Rightarrow w_{+} - w_{-} = i(v_{+} - v_{-}) = f \text{ on } \Gamma$$

$$\Rightarrow f \text{ pure imaginary on } \Gamma$$

$$\Rightarrow h \text{ real on } \Gamma \quad (\because \int_{-1}^1 \frac{f(\xi)}{\xi-x} d\xi \text{ real on } \Gamma)$$

$$\text{Add } \Rightarrow w_+ + w_- = 2g + i(v_+ + v_-) = 2h \text{ on } \mathbb{P}$$

$$\Rightarrow v_+ + v_- = 0, w_+ + w_- = 2g \text{ on } \mathbb{P}$$

Hence,

$$F = W_+ - W_- = \frac{w_+}{\tilde{w}_+} - \frac{w_-}{\tilde{w}_-} = \frac{w_+ + w_-}{\tilde{w}_+} = \frac{2g}{\tilde{w}_+} \text{ on } \mathbb{P} \quad \square$$

Moreover,

$$f = w_+ - w_- = \tilde{w}_+ W_+ - \tilde{w}_- W_- = \tilde{w}_+ (W_+ + W_-) \text{ on } \mathbb{P}$$

$$\Rightarrow f(x) = 2\tilde{w}_+(x)H(x)$$

$$= \frac{\tilde{w}_+(x)}{\pi i} \int_{-1}^1 \frac{F(\zeta)}{\zeta - x} d\zeta$$

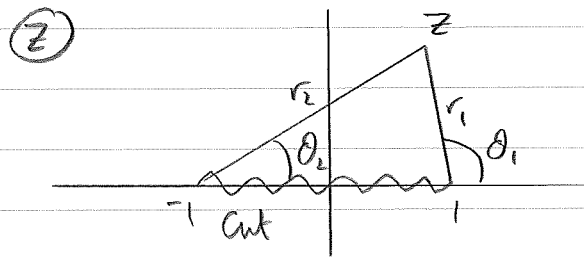
$$= \frac{2\tilde{w}_+(x)}{\pi i} \int_{-1}^1 \frac{g(\zeta)}{\tilde{w}_+(\zeta)(\zeta - x)} d\zeta \quad (|x| < 1)$$

satisfies

$$2g = w_+ + w_- = 2h \text{ on } \mathbb{P}$$

$$\text{i.e. } g(x) = \frac{1}{2\pi i} \int_{-1}^1 \frac{f(\zeta)}{\zeta - x} d\zeta \quad (|x| < 1) \quad \square$$

2(c) Define $w(z) = \log\left(\frac{z-1}{z+1}\right) = \log\left(\frac{r_1}{r_2}\right) + i(\theta_1 - \theta_2)$, where $z-1 = r_1 e^{i\theta_1}$, $z+1 = r_2 e^{i\theta_2}$; $r_1, r_2 > 0$; $\theta_1, \theta_2 \in (-\pi, \pi]$.



$$\Rightarrow w_{\pm}(x) = \log\left(\frac{1-x}{1+x}\right) \pm i\pi \text{ for } |x| < 1$$

$$(PF) \Rightarrow f(x) = w_{+}(x) - w_{-}(x) = 2i\pi \text{ for } |x| < 1$$

$$\text{and } \frac{1}{\pi i} \int_{-1}^1 \frac{f(t)}{t-x} dt = w_{+}(x) + w_{-}(x) = 2\log\left(\frac{1-x}{1+x}\right) \text{ for } |x| < 1$$

$$\text{Hence, } \frac{1}{\pi i} \int_{-1}^1 \frac{2i\pi}{t-x} dt = 2\log\left(\frac{1-x}{1+x}\right) \text{ for } |x| < 1$$

$$\Rightarrow \int_{-1}^1 \frac{dt}{t-x} = \log\left(\frac{1-x}{1+x}\right) \text{ for } |x| < 1$$

2(d) Repeat calculation with $w(z) = \log^2\left(\frac{z-1}{z+1}\right)$

$$\Rightarrow f(x) = \left(\log\left(\frac{1-x}{1+x}\right) + i\pi\right)^2 - \left(\log\left(\frac{1-x}{1+x}\right) - i\pi\right)^2 \text{ for } |x| < 1$$

$$\text{and } \frac{1}{\pi i} \int_{-1}^1 \frac{f(t)}{t-x} dt = \left(\log\left(\frac{1-x}{1+x}\right) + i\pi\right)^2 + \left(\log\left(\frac{1-x}{1+x}\right) - i\pi\right)^2 \text{ for } |x| < 1$$

$$\Rightarrow \frac{1}{\pi i} \int_{-1}^1 4i\pi \log\left(\frac{1-t}{1+t}\right) \frac{dt}{t-x} = 2\log^2\left(\frac{1-x}{1+x}\right) - 2\pi^2 \text{ for } |x| < 1$$

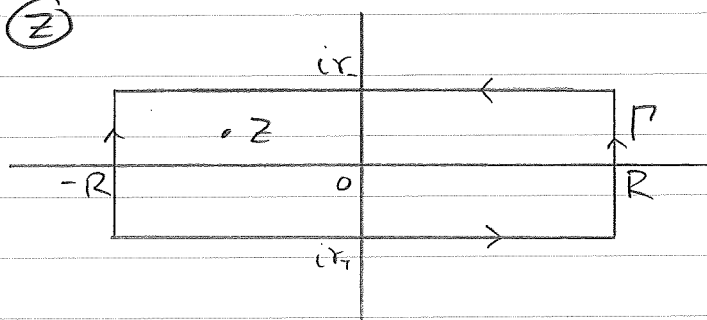
$$\Rightarrow \int_{-1}^1 \log\left(\frac{1-t}{1+t}\right) \frac{dt}{t-x} = \frac{1}{2} \log^2\left(\frac{1-x}{1+x}\right) - \frac{1}{2} \pi^2 \text{ for } |x| < 1$$

3(a1) Since $w_+ = w_-$ in $\alpha < \text{Im}(z) < \beta$, w_- is the analytic continuation of w_+ into the lower-half plane $\text{Im}(z) \leq \alpha$, and together they define an entire function, $E(z)$ say.

w_{\pm} bounded $\Rightarrow E$ bounded

E entire and bounded $\Rightarrow E = w_+ = w_-$ is constant by Liouville's theorem.

(a2) Fix δ_{\pm} s.t. $\alpha < \delta_+ < \delta_- < \beta$ and consider the rectangular contour Γ :



For z inside Γ , Cauchy's formula $\Rightarrow G(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{G(\zeta)}{\zeta - z} d\zeta$.

$G(z) \rightarrow 0$ as $|z| \rightarrow \infty$ with $\alpha < \text{Im}(z) < \beta \Rightarrow$ contribution from vertical sides tends to zero as $R \rightarrow \infty \Rightarrow$

$$G(z) = G_+(z) - G_-(z) \text{ for } \delta_+ < \text{Im}(z) < \delta_-$$

where $G_{\pm}(z) = \frac{1}{2\pi i} \int_{\Gamma_{\pm}} \frac{G(\zeta)}{\zeta - z} d\zeta$, $\Gamma_{\pm} = \{\alpha + i\delta_{\pm} : -\infty < x < \infty\}$.

Now, G_{\pm} holomorphic on $\mathbb{C} \setminus \Gamma_{\pm}$, i.e. $G_{\pm} \in H(\mathbb{C} \setminus \Gamma_{\pm})$
 $\Rightarrow G_+ \in H(\text{Im}(z) > \delta_+)$ and $G_- \in H(\text{Im}(z) < \delta_-)$.

Thus, $w_+ - G_+$ is holomorphic and bounded at ∞ in $\text{Im}(z) > \delta_+$,
 $w_- - G_-$ is holomorphic and bounded at ∞ in $\text{Im}(z) < \delta_-$ and
 $w_+ - G_+ = w_- - G_-$ in $\delta_+ < \text{Im}(z) < \delta_-$, so by (a1),
 $w_+ - G_+ = w_- - G_-$ is a constant. □

3(iii) Since $g(z) = \log F(z)$ satisfies the conditions in (aii),

$$\log F(z) = g_+(z) - g_-(z) \text{ for } \delta_+ < \text{Im}(z) < \delta_-.$$

Thus, define $F_{\pm}(z) = e^{g_{\pm}(z)}$ to obtain

$$F(z) = e^{g_+(z) - g_-(z)} = \frac{F_+(z)}{F_-(z)} \text{ for } \delta_+ < \text{Im}(z) < \delta_-,$$

where

$$F_{\pm}(z) = \exp \left[\frac{1}{2\pi i} \int_{\Gamma_{\pm}} \frac{\log F(z)}{z-z} dz \right] \text{ for } \delta_+ < \text{Im}(z) < \delta_-.$$

Thus, $F_+ w_+$ is holomorphic and bounded at ∞ in $\text{Im}(z) > \delta_+$,
 $w_- F_-$ is holomorphic and bounded at ∞ in $\text{Im}(z) < \delta_-$,
 and $w_+ F_+ = w_- F_-$ in $\delta_+ < \text{Im}(z) < \delta_-$, so by (ai),
 $w_+ F_+ = w_- F_-$ is a constant. \square

(bi) $\bar{u}(k, y) = \int_{-\infty}^{\infty} u(x, y) e^{ikx} dx \Rightarrow \bar{u}_{yy} - (k^2 + 1)\bar{u} = 0$ in $y > 0$.

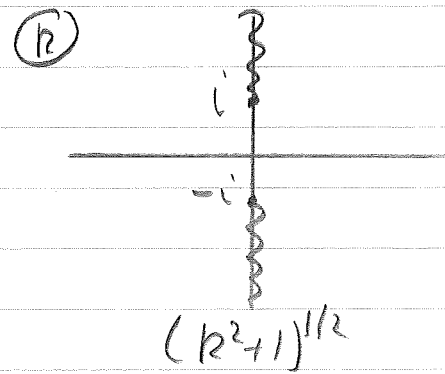
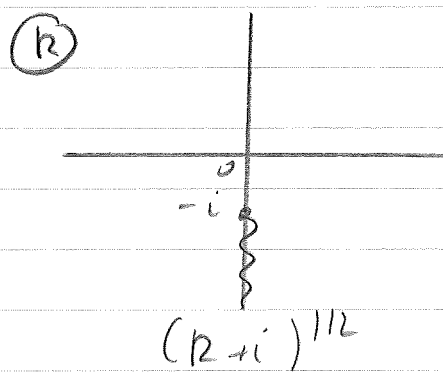
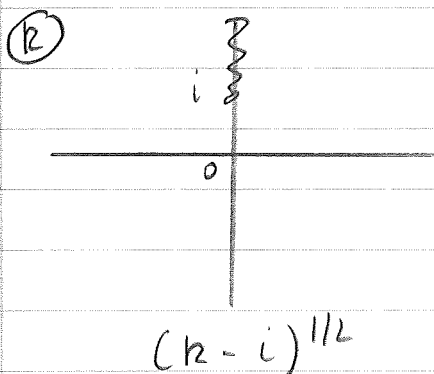
$$\bar{u} \rightarrow 0 \text{ as } y \rightarrow \infty \Rightarrow \bar{u} = A(k) e^{-(k^2+1)^{1/2} y}$$

where the branch of $(k^2+1)^{1/2}$ must be chosen so that $\text{Re}(k^2+1)^{1/2} > 0$ on the inversion contour.

$$\begin{aligned} \text{Then, define } (k^2+1)^{1/2} &:= (k-i)^{1/2} (k+i)^{1/2} \\ (k-i)^{1/2} &:= r_1^{1/2} e^{i\theta_1/2} \\ (k+i)^{1/2} &:= r_2^{1/2} e^{i\theta_2/2} \end{aligned}$$

where $r_1 = |k-i| > 0$, $r_2 = |k+i| > 0$, $\theta_1 = \arg(k-i) \in (-\pi/2, \pi/2]$

and $\theta_2 = \arg(k+i) \in (-\frac{\pi}{2}, \frac{3\pi}{2}] \Rightarrow$ following branch cuts



• $f(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases} \Rightarrow \bar{f}(k) = \int_0^{\infty} e^{ikx} dx = \frac{i}{k} \text{ for } \text{Im}(k) > 0$

$f(x) = O(e^{bx})$ as $x \rightarrow \infty \Rightarrow \bar{f} \in H(\text{Im}(k) < b)$

Thus, $\bar{u}(k, 0) = \bar{f}_-(k) + \frac{i}{k}$ for $0 < \text{Im}(k) < b$.

• $g_+(x) = O(e^{ax})$ as $x \rightarrow \infty \Rightarrow \bar{g}_+ \in H(\text{Im}(k) > a)$

Thus, $\bar{u}_y(k, 0) = \bar{g}_+(k)$ for $\text{Im}(k) > a$.

• Eliminate $A(k)$: $\bar{g}_+(k) = -(k^2+1)^{1/2} A(k)$ for $\text{Im}(k) > a$
 $\bar{f}_-(k) + \frac{i}{k} = A(k)$ for $0 < \text{Im}(k) < b$

• Since $a \leq 0, b \geq 1$, we obtain

$$\bar{g}_+(k) + (k^2+1)^{1/2} \left(\bar{f}_-(k) + \frac{i}{k} \right) = 0 \text{ for } 0 < \text{Im}(k) < 1$$

(bii) Split $(z^2+1)^{1/2} \Rightarrow \frac{\bar{g}_+(z)}{(z+i)^{1/2}} + (z-i)^{1/2} \bar{f}_-(z) = -\frac{i(z-i)^{1/2}}{z}$ for $0 < \text{Im}(z) < 1$

Split RHS $\Rightarrow \frac{(z-i)^{1/2}}{z} = \frac{(z-i)^{1/2} - (0-i)^{1/2}}{z} + \frac{(0-i)^{1/2}}{z}$

where $(0-i)^{1/2} = e^{-i\pi/4}$ because $r_1 = 1, \theta_1 = -\frac{\pi}{2}$.

Hence,

$$\frac{\bar{g}_+(z)}{(z+i)^{1/2}} + \frac{i(0-i)^{1/2}}{z} = -(z-i)^{1/2} \bar{f}_-(z) + \frac{i}{z} ((0-i)^{1/2} - (z-i)^{1/2}) \quad (\#)$$

for $0 < \text{Im}(z) < 1$.

- LHS (#) $\in H(\text{Im}(z) > 0)$, RHS (#) $\in H(\text{Im}(z) < 1)$;
LHS (#) $\rightarrow 0$ as $|z| \rightarrow \infty$ with $\text{Im}(z) > 0$, RHS (#) $\rightarrow 0$ as $|z| \rightarrow \infty$ with $\text{Im}(z) < 1$.

Hence, by (ai), LHS (#) = RHS (#) = constant = 0

$$\Rightarrow \bar{f}_-(z) = \frac{i}{(z-i)^{1/2} z} ((0-i)^{1/2} - (z-i)^{1/2}), \quad \bar{g}_+(z) = \frac{-i(0-i)^{1/2}(z+i)^{1/2}}{z}$$

(biii) Γ^+ (Γ^-) must pass above (below) the singularities of $\bar{f}_+(z)$ ($\bar{g}_-(z)$), e.g. $\Gamma^+ \subseteq \{\text{Im}(z) > 0\}$ and $\Gamma^- \subseteq \{\text{Im}(z) < 1\}$

