2B68

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C6.3b

APPLIED COMPLEX VARIABLES

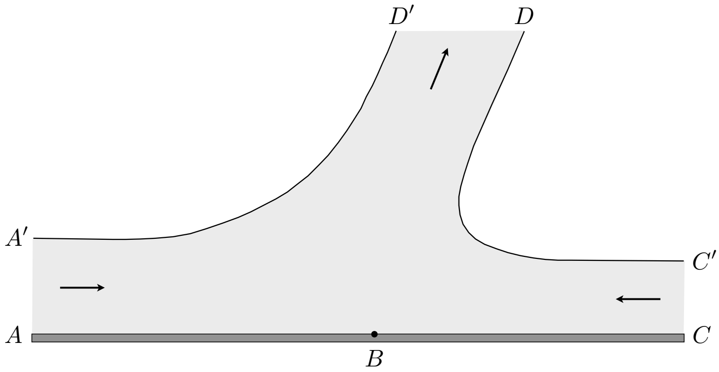
TRINITY TERM 2013

THURSDAY, 30 MAY 2013, 2.30pm to 4.00pm

You may submit answers to as many questions as you wish; the best two will count for the total mark.
You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

Do not turn this page until you are told that you may do so

1. Consider the steady two-dimensional potential flow illustrated below in which two impacting jets eject a third jet above a rigid straight wall AC. The arrows indicate the far-field fluid velocity in the jets.



The points A, A', C and C' lie at $(-\infty, 0)$, $(-\infty, 1)$, $(\infty, 0)$ and (∞, h) respectively, where $h > 0$. The fluid velocity (u, v) is equal to $(1, 0)$, $(-1, 0)$ and $(\cos\alpha, \sin\alpha)$ in the far field of the jets at the points A, C and D respectively, where $0 < \alpha < \pi$. The point B is a stagnation point at which the potential $\phi = 0$. On the free surfaces $A'D'$ and $C'D$, $u^2 + v^2 = 1$. Take the stream function $\psi = 0$ on $C'D$, so that $\psi = h$ on AC and $\psi = 1 + h$ on $A'D'$.

(a) Sketch the potential plane ($w = \phi + \mathrm{i}\psi$), showing that it is a strip with a semi-infinite slit removed. Sketch the hodograph plane ($w' = u - \mathrm{i}v$), showing that it is a semicircle. In your sketches label clearly the locations of the points A, A', B, C, C', D and D'.

(b) Show that the Schwarz–Christoffel map from the upper half Z-plane to the potential plane, with $Z = 0$, $Z = \beta$, $Z = 1$ and $Z = \infty$ mapped to A, B, C and D respectively, is

$$\pi w = \log Z + h \log(Z - 1) + (1 + h)\log(1 + h) - h \log h,$$

where log denotes the principal branch of the logarithm. You should determine an expression for β in terms of h.

(c) Show that the map from the hodograph plane to the upper half ζ-plane taking A to $\zeta = 0$, C to $\zeta = 1$ and D to $\zeta = \infty$ is

$$\frac{1}{\zeta} = 1 + \left(\frac{1 + w'}{1 - w'}\right)^2 \tan^2\left(\frac{\alpha}{2}\right).$$

(d) Determine an expression for h in terms of α. Sketch the flow domain for the cases when (i) $h \ll 1$, (ii) $h = 1$ and (iii) $h \gg 1$.

2. (a) Show that if Γ is a contour in the complex plane and

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z},$$

then, if f is continuous on Γ, t is any point at which Γ is smooth and f is holomorphic in a neighbourhood of t, the limiting values of $w(z)$ as Γ is approached from either side are $w_\pm(t)$, where

$$w_\pm(t) = \pm\frac{1}{2} f(t) + \frac{1}{2\pi \mathrm{i}} ⨍_\Gamma \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - t}.$$

You should define the integral $⨍$ precisely.

(b) Let $\Gamma = \{x + \mathrm{i}y : |x| < 1, y = 0\}$ and $\overline{\Gamma} = \{x + \mathrm{i}y : |x| \le 1, y = 0\}$. Suppose $w(z)$ is holomorphic away from $\overline{\Gamma}$ and $a(x)w_+(x) + b(x)w_-(x) = c(x)$ on Γ for some known smooth complex-valued functions $a(x)$, $b(x)$ and $c(x)$, where $a(x)$ and $b(x)$ are non-zero for $|x| < 1$. Suppose $\widetilde{w}(z)$ is holomorphic and non-zero away from $\overline{\Gamma}$ and $a(x)\widetilde{w}_+(x) = -b(x)\widetilde{w}_-(x) \neq 0$ on Γ. Determine the density $F(\xi)$ for which a solution for $w(z)$ is

$$\frac{w(z)}{\widetilde{w}(z)} = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{F(\xi)\mathrm{d}\xi}{\xi - z}.$$

Deduce that

$$f(x) = \frac{(b(x) - a(x))c(x)}{2a(x)b(x)} + \frac{(a(x) + b(x))\widetilde{w}_+(x)}{2\pi \mathrm{i} b(x)} ⨍_{-1}^{1} \frac{c(\xi)\,\mathrm{d}\xi}{a(\xi)\widetilde{w}_+(\xi)(\xi - x)}$$

is a solution of the singular integral equation

$$\frac{(a(x) - b(x))}{2} f(x) + \frac{(a(x) + b(x))}{2\pi \mathrm{i}} ⨍_{-1}^{1} \frac{f(\xi)\,\mathrm{d}\xi}{\xi - x} = c(x) \quad \text{for } |x| < 1.$$

(c) By taking $\int_{-1}^{1} \xi f(\xi)\,\mathrm{d}\xi$ to be a constant that you should determine, deduce that

$$f(x) = \frac{2x}{\pi(2 + \alpha)(1 - x^2)^{1/2}}$$

is a solution of the singular integral equation

$$⨍_{-1}^{1} \left(\frac{1}{\xi - x} + \alpha\xi \right) f(\xi)\,\mathrm{d}\xi = 1 \quad \text{for } |x| < 1$$

provided that $\alpha \neq -2$.

[*You may use without proof the fact that* $⨍_{-1}^{1} \frac{(1-\xi^2)^{1/2}}{x-\xi}\,\mathrm{d}\xi = \pi x$ *for* $|x| < 1$.]

3. (a) Suppose that

$$\nabla^2 u = a^2 u \quad \text{in } y > 0,$$

with

$$u = \mathrm{e}^{ibx} \quad \text{on } y = 0,\ x > 0, \qquad \frac{\partial u}{\partial y} = 0 \quad \text{on } y = 0,\ x < 0,$$

and $u \to 0$ as $x^2 + y^2 \to \infty$, where a and b are positive constants. Define

$$f_-(x) = \begin{cases} u(x,0) & \text{for } x < 0, \\ 0 & \text{for } x > 0, \end{cases} \qquad g_+(x) = \begin{cases} 0 & \text{for } x < 0, \\ \dfrac{\partial u}{\partial y}(x,0) & \text{for } x > 0. \end{cases}$$

Suppose that $\mathrm{e}^{-ax} f_-(x) \to 0$ as $x \to -\infty$ and $g_+(x)$ is bounded as $x \to \infty$.

(i) By taking a Fourier transform show that

$$\overline{g}_+(k) + (k^2 + a^2)^{1/2} \left(\overline{f}_-(k) + \frac{\mathrm{i}}{k+b} \right) = 0 \quad \text{for } 0 < \mathrm{Im}(k) < a, \tag{1}$$

where you should define precisely the branch of $(k^2 + a^2)^{1/2}$.

(ii) Deduce from (1) expressions for $\overline{f}_-(k)$ and $\overline{g}_+(k)$, defining precisely the branch of each multi-valued function that you use.

[*You may use without proof the fact that* $k^{1/2}\overline{g}_+(k)$ *and* $k\overline{f}_-(k)$ *are bounded as* $|k| \to \infty$.]

(b) Suppose that $h(x) = \mathrm{e}^{c|x|} + \mathrm{e}^{-|x|}$, where c is a positive constant. Let

$$h_-(x) = \begin{cases} h(x) & \text{for } x < 0, \\ 0 & \text{for } x > 0, \end{cases} \qquad h_+(x) = \begin{cases} 0 & \text{for } x < 0, \\ h(x) & \text{for } x > 0. \end{cases}$$

(i) Determine the Fourier transforms $\overline{h}_\pm(k)$, stating clearly where the Fourier integrals converge.

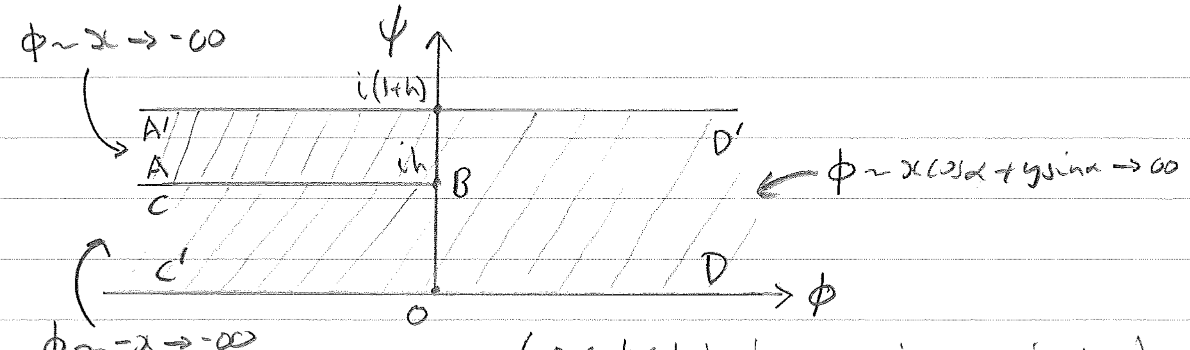
(ii) To which parts of the complex k-plane may $\overline{h}_\pm(k)$ be analytically continued? Over which part of the complex k-plane is it possible to define $\overline{h}(k)$?

(iii) Sketch a suitable inversion contour Γ for which

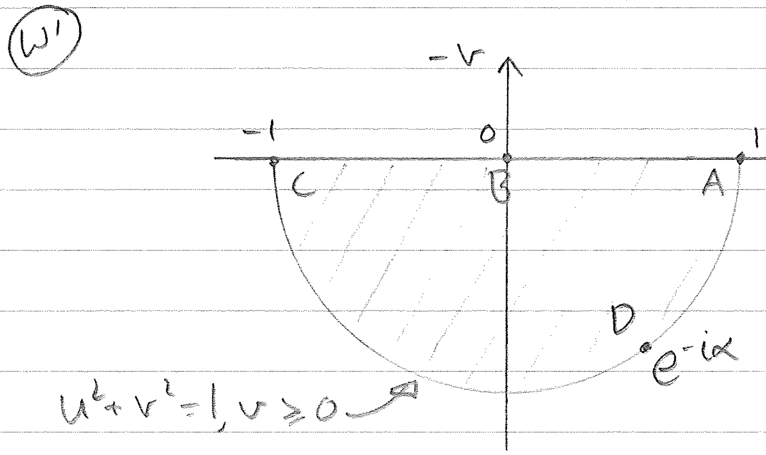
$$h(x) = \frac{1}{2\pi} \int_\Gamma \overline{h}(k) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}k$$

when $0 < c < 1$ and when $c > 1$.

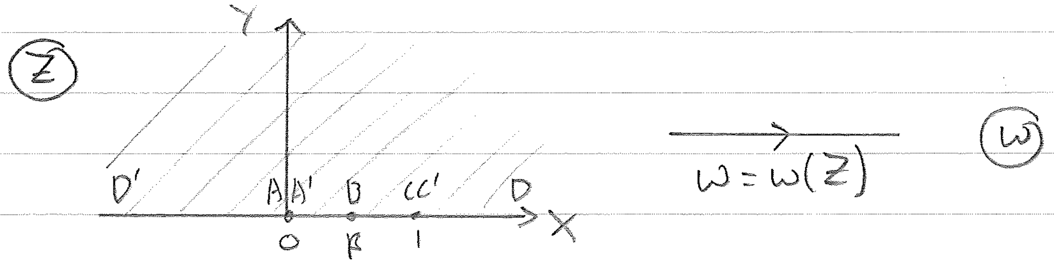
1(a)



(0 < ψ < 1+h by maximum principle)



(b)



Schwarz-Christoffel formula with $n=4$ vertices and exterior angles $\beta_A = 1$, $\beta_B = -1$, $\beta_C = 1$, $\beta_D = 1$

$$\Rightarrow \quad \frac{dw}{dZ} = P\,(Z - Z_A)^{-\beta_A}(Z - Z_B)^{-\beta_B}(Z - Z_C)^{-\beta_C}$$

where $P \in \mathbb{C}$ and we choose $Z_A = 0$, $Z_B = \beta$, $Z_C = 1$, $Z_D = \infty$ as instructed, with $0 < \beta < 1$ to preserve orientation of points.

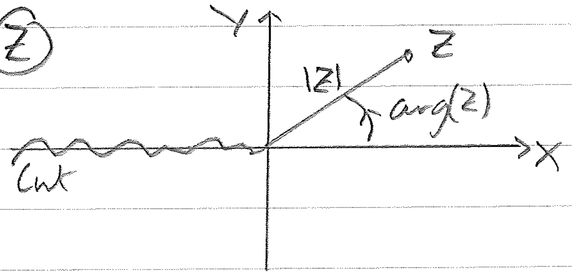
$$\Rightarrow \quad \frac{dw}{dZ} = P\,\frac{(Z-\beta)}{Z(Z-1)} = P\left[\frac{\beta}{Z} + \frac{1-\beta}{Z-1}\right]$$

$$\Rightarrow \quad w = P\left[\beta \log Z + (1-\beta)\log(Z-1)\right] + Q$$

where $Q \in \mathbb{C}$ and we choose the principal branch of the logarithm, with

$$\log z := \log|z| + i\arg(z) \quad , \quad -\pi < \arg(z) \le \pi$$

and branch cut as shown: z



$\boxed{4}$ Bookwork

$C'D : \psi = 0, \; z = x > 1 \Rightarrow P \in \mathbb{R}^+, \; Q \in \mathbb{R}$

$B \; : \; ih = P\left[\beta\log\beta + (1-\beta)\left(\log(1-\beta) + i\pi\right)\right] + Q$

$$\Rightarrow h = \pi P(1-\beta), \quad Q = -P\left[\beta\log\beta + (1-\beta)\log(1-\beta)\right]$$

$A'D' : (1+h)i = \psi i = P\left[\beta\pi i + (1-\beta)\pi i\right]$

$$\Rightarrow \pi P = 1+h$$

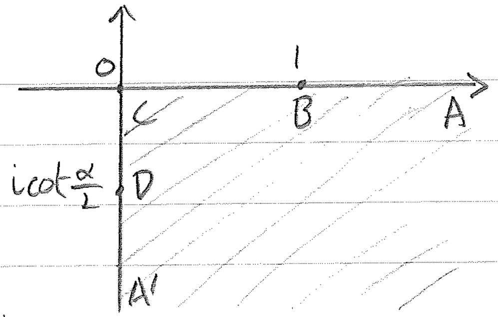
Hence, $h = \pi P(1-\beta) = (1-\beta)(1+h) \Rightarrow \beta = 1 - \frac{h}{1+h} = \frac{1}{1+h}$ and

$$\pi w = (1+h)\left[\frac{1}{1+h}\log z + \left(1 - \frac{1}{1+h}\right)\log(z-1) - \frac{1}{1+h}\log\left(\frac{1}{1+h}\right) - \left(1 - \frac{1}{1+h}\right)\log\left(1 - \frac{1}{1+h}\right)\right]$$

$$= \log z + h\log(z-1) - \log\left(\frac{1}{1+h}\right) - h\log\frac{h}{1+h}$$

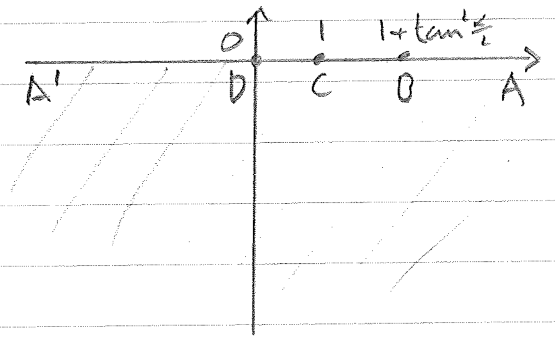
$$= \log z + h\log(z-1) + (1+h)\log(1+h) - h\log h \qquad \square$$

(c) $$w' \xrightarrow{\quad z_1 = \frac{1+w'}{1-w'} \quad} z_1$$

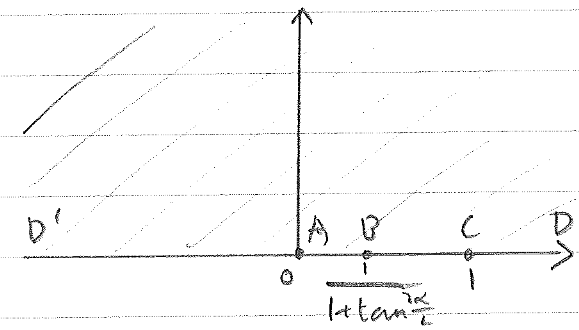


$$1 \to \infty;\ 0 \to 1;\ -1 \to 0;\ e^{-i\alpha} \to \frac{1+e^{-i\alpha}}{1-e^{-i\alpha}} = \frac{2\cos\frac{\alpha}{2}}{2i\sin\frac{\alpha}{2}} = -i\cot\frac{\alpha}{2}$$

$$z_1 \xrightarrow{\quad z_2 = 1 + z_1^2\tan^2\frac{\alpha}{2} \quad} z_2$$



$$z_2 \xrightarrow{\quad z = 1/z_2 \quad} z$$



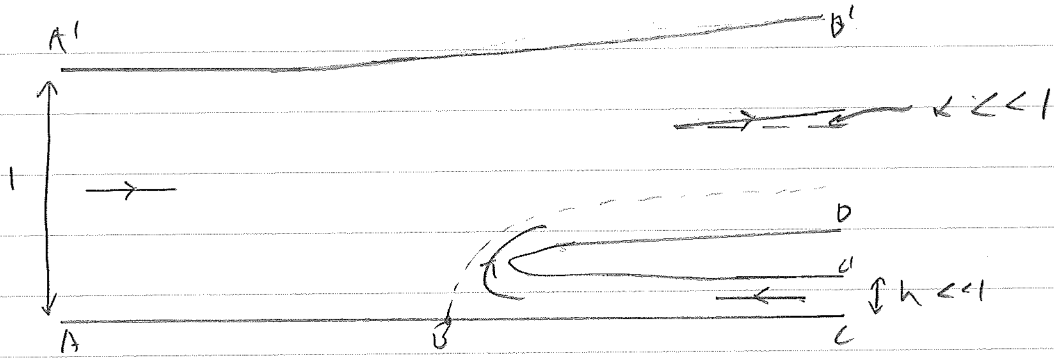
Thus, $\frac{1}{z} = z_2 = 1 + z_1^2\tan^2\frac{\alpha}{2} = 1 + \left(\frac{1+w'}{1-w'}\right)^2\tan^2\frac{\alpha}{2}$ $\square$

(d) Same points A, C, D in Z- and z-planes $\Rightarrow Z = z$

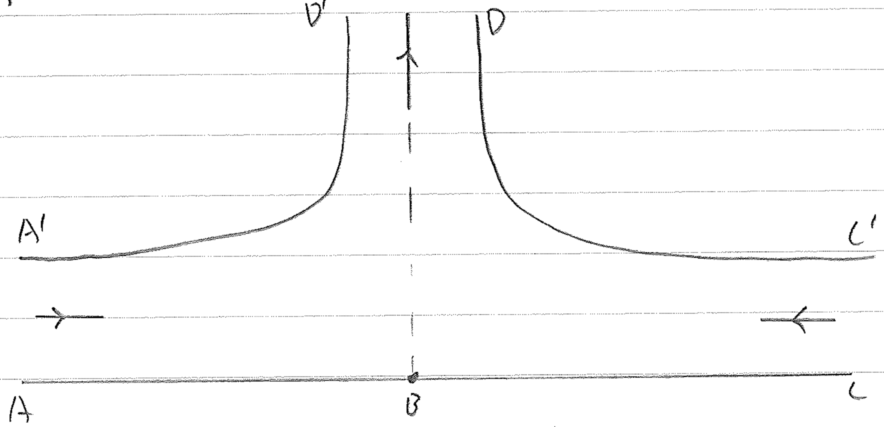
$\Rightarrow$ same point B, i.e. $\beta = \frac{1}{1+h} = \frac{1}{1+\tan^2\frac{\alpha}{2}}$

$\Rightarrow h = \tan^2\frac{\alpha}{2}$.

(i) $h \ll 1$

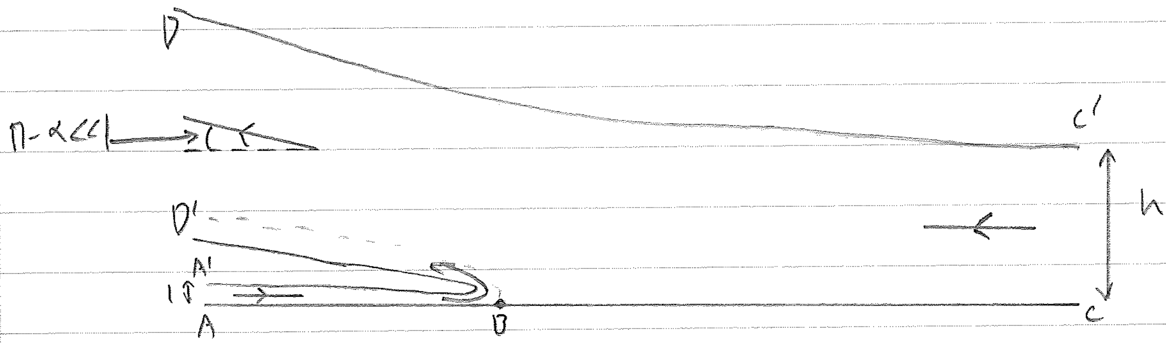


(ii) $h = 1$



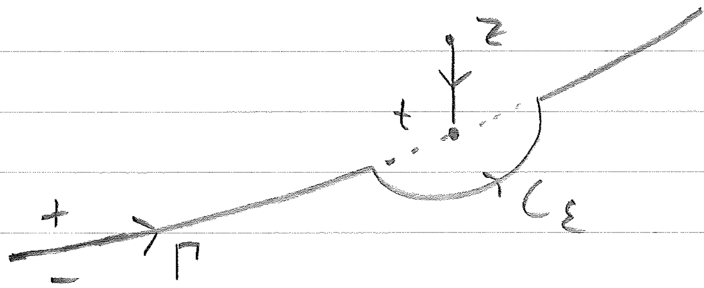
Symmetric about $x = 0$.

(iii) $h \gg 1$



2(a) Label the LHS of Γ as "+" and the RHS as "−", oriented in the direction of integration.

As $z \to t \in \Gamma$ from the + side indent Γ with a small (approximate) semi-circle C_ε around t as shown, where the radius ε is sufficiently small that f is holomorphic in $D(t, 2\varepsilon) = \{z : |z-t| < 2\varepsilon\}$



Let $\gamma_\varepsilon = \Gamma \cap D(t,\varepsilon)$, i.e. the portion of Γ replaced by C_ε.

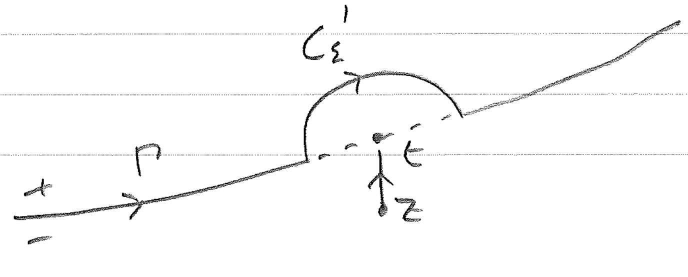
Deformation thm $\Rightarrow$ $$w(z) = \frac{1}{2\pi i}\left(\int_{\Gamma\setminus\gamma_\varepsilon} + \int_{C_\varepsilon}\right)\frac{f(\zeta)}{\zeta - z}\,d\zeta$$

$$\underset{(z\to t)}{\Rightarrow} \quad w_+(t) = \frac{1}{2\pi i}\left(\int_{\Gamma\setminus\gamma_\varepsilon} + \int_{C_\varepsilon}\right)\frac{f(\zeta)}{\zeta - t}\,d\zeta$$

$$\underset{(\varepsilon\to 0)}{\Rightarrow} \quad w_+(t) = \underbrace{\frac{1}{2\pi i}⨍_{\Gamma}\frac{f(\zeta)}{\zeta - t}\,d\zeta}_{⨍_\Gamma := \lim_{\varepsilon\to 0}\int_{\Gamma\setminus\gamma_\varepsilon}} + \underbrace{\frac{1}{2}\cdot\frac{2\pi i}{2\pi i}\operatorname*{res}_{\zeta = t}\frac{f(\zeta)}{\zeta - t}}_{= \frac{1}{2}f(t)}$$

The PVI exists $\because$ log singularities cancel as $\varepsilon \to 0$ by cty.

For $w_-(t)$ replace γ_ε with C'_ε as shown:



Semi-circle now gives a contribution

$$-\frac{1}{2}\frac{2\pi i}{2\pi i}\operatorname*{res}_{\zeta=t}\frac{f(\zeta)}{\zeta-t} = -\frac{1}{2}f(t)$$

Hence,

$$w_\pm(t) = \pm\frac{1}{2}f(t) + \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta-t}\,d\zeta \qquad \text{(PF)}$$

$\square$

2(b) Seek a solution by setting

$$w(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{f(\zeta)}{\zeta-z}\,d\zeta, \quad \frac{w(z)}{\tilde{w}(z)} = W(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{F(\zeta)}{\zeta-z}\,d\zeta$$

where the densities f and F are TBD.

$$\text{(PF)} \Rightarrow \quad w_\pm = \pm\frac{1}{2}f + h, \quad W_\pm = \pm\frac{1}{2}F + H \quad \text{on } \Gamma,$$

where $h(x) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{f(\zeta)}{\zeta-x}\,d\zeta$, $H(x) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{F(\zeta)}{\zeta-x}\,d\zeta$ $(|x|<1)$

On Γ,
$$F = W_+ - W_-$$
$$= \frac{w_+}{\tilde{w}_+} - \frac{w_-}{\tilde{w}_-}$$
$$= \frac{w_+}{\tilde{w}_+} - \frac{w_-}{-a\tilde{w}_+/b}$$
$$= \frac{aw_+ + bw_-}{a\tilde{w}_+}$$
$$= \frac{c}{a\tilde{w}_+}$$

$\square$

On Γ, $f = w_+ - w_-$

$$= \tilde{w}_+ W_+ - \tilde{w}_- W_-$$

$$= \tilde{w}_+ W_+ - \left(-\frac{a\tilde{w}_+}{b}\right) W_-$$

$$= \frac{\tilde{w}_+}{b}\left(aW_- + bW_+\right)$$

$$= \frac{\tilde{w}_+}{b}\left(\frac{b-a}{2}F + (a+b)H\right)$$

$$= \frac{(b-a)c\,\tilde{w}_+}{2ab\,\tilde{w}_+} + \frac{(a+b)\tilde{w}_+ H}{b}$$

$$\Rightarrow \quad f(x) = \frac{(b(x)-a(x))\,c(x)}{2a(x)\,b(x)} + \frac{(a(x)+b(x))\,\tilde{w}_+(x)}{2\pi i\, b(x)} \int_{-1}^{1} \frac{c(\xi)\,d\xi}{a(\xi)\,\tilde{w}_+(\xi)(\xi - x)} \quad (|x|<1)$$

satisfies

$$a w_+ + b w_- = c \quad \text{on } \Gamma$$

i.e. $a\left(+\tfrac{1}{2}f + h\right) + b\left(-\tfrac{1}{2}f + h\right) = c$ on Γ

i.e. $\tfrac{1}{2}(a-b)f + (a+b)h = c$ on Γ

i.e. $$\frac{(a(x)-b(x))}{2} f(x) + \frac{(a(x)+b(x))}{2\pi i} \int_{-1}^{1} \frac{f(\xi)}{\xi - x}\, d\xi = c(x) \quad \text{for } |x|<1$$

$\square$

2(c) Let $A = \int_{-1}^{1} \zeta f(\zeta)\,d\zeta \Rightarrow \fint_{-1}^{1} \frac{f(\zeta)}{\zeta - x}\,d\zeta = 1 - \alpha A$ for $|x|<1$

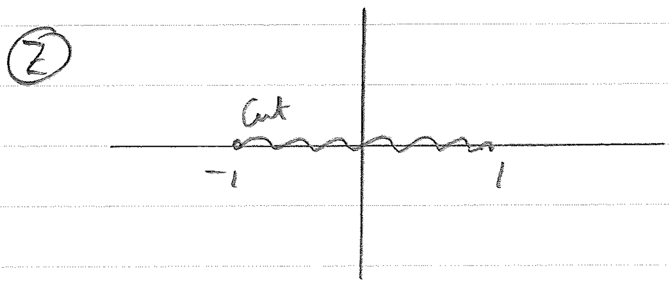
Hence, in 2(b) let $a = b = 1$, $\pi i c = 1 - \alpha A$

$$\Rightarrow f(x) = \frac{\tilde{\omega}_+(x)}{\pi i} \fint_{-1}^{1} \frac{(1-\alpha A)/\pi i}{\tilde{\omega}_+(x)(\zeta - x)}\,d\zeta \text{ is a solution.}$$

Since $a = b = 1$, choose

$$\tilde{\omega}(z) = (z^2-1)^{-1/2} = |z^2-1|^{-1/2} e^{-i(\arg(z-1)+\arg(z+1))/2}$$

where $-\pi < \arg(z \pm 1) \le \pi$, so that branch cut is as shown:



Thus, $\tilde{\omega}_+(x) = -i(1-x^2)^{-1/2}$ for $|x|<1$

$$\Rightarrow f(x) = \frac{-i(1-x^2)^{-1/2}}{\pi i} \fint_{-1}^{1} \frac{(1-\alpha A)(1-\zeta^2)^{1/2}}{\pi i(-i)(\zeta - x)}\,d\zeta$$

$$= -\frac{(1-\alpha A)}{\pi^2(1-x^2)^{1/2}} \underbrace{\fint_{-1}^{1} \frac{(1-\zeta^2)^{1/2}}{\zeta - x}\,d\zeta}_{= -\pi x \text{ by hint}}$$

$$= \frac{(1-\alpha A)x}{\pi(1-x^2)^{1/2}}$$

$$\Rightarrow A = \frac{(1-\alpha A)}{\pi} \int_{-1}^{1} \frac{x^2\,dx}{(1-x^2)^{1/2}} \underset{x = \sin\theta}{\uparrow=} \frac{(1-\alpha A)}{\pi} 2 \underbrace{\int_0^{\pi/2} \sin^2\theta\,d\theta}_{= 2\pi/4} = \frac{1}{2}(1-\alpha A)$$

$\Rightarrow A = \frac{1}{2+\alpha}$, $1 - \alpha A = \frac{2}{2+\alpha}$ provided $\alpha \neq -2$

$$\Rightarrow f(x) = \frac{2x}{\pi(2+\alpha)(1-x^2)^{1/2}}$$

□

3(a)(i) $\bar{u}(k,y) = \int_{-\infty}^{\infty} u(x,y)e^{ikx}\,dx \Rightarrow \bar{u}_{yy} - (k^2+a^2)\bar{u} = 0$ in $y>0$

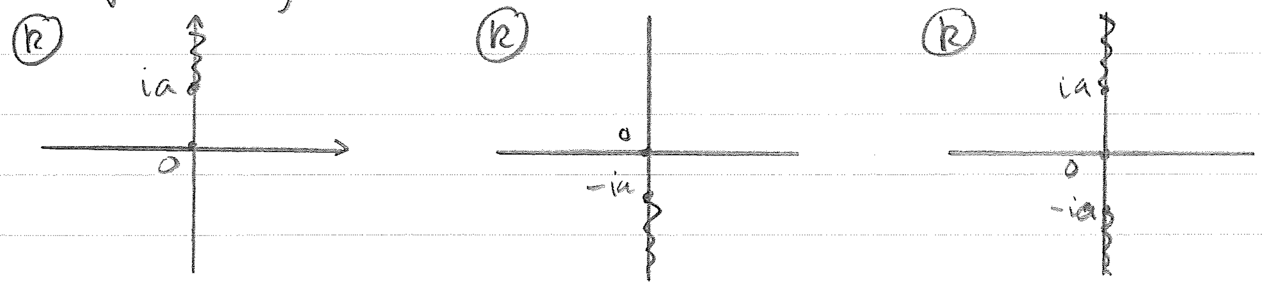
and $\bar{u} \to 0$ as $y \to \infty \Rightarrow \bar{u} = A(k)e^{-(k^2+a^2)^{1/2}y}$,

where the branch of $(k^2+a^2)^{1/2}$ must be chosen so that $\mathrm{Re}(k^2+a^2)^{1/2} > 0$ on the inversion contour.

- Thus, define $(k^2+a^2)^{1/2} = (k-ia)^{1/2}(k+ia)^{1/2}$ $(a>0)$
  $(k \pm ia)^{1/2} = |k \pm ia|^{1/2} e^{i\arg(k \pm ia)/2}$

where $\arg(k-ia) \in (-\frac{3\pi}{2}, \frac{\pi}{2}]$, $\arg(k+ia) \in (-\frac{\pi}{2}, \frac{3\pi}{2}]$

$\Rightarrow$ following branch cuts:



$(k-ia)^{1/2}$ $\quad$ $(k+ia)^{1/2}$ $\quad$ $(k^2+a^2)^{1/2}$

- Now, $\int_0^{\infty} e^{ibx}e^{ikx}\,dx = \left.\frac{e^{i(b+k)x}}{i(b+k)}\right|_0^{\infty} = \frac{i}{b+k}$ for $\mathrm{Im}(k)>0$

and $e^{-ax}f_-(x) \to 0$ as $x \to -\infty \Rightarrow \bar{f} \in H(\mathrm{Im}(k) < a)$.

Thus, $A(k) = \bar{u}(k,0) = \bar{f}_-(k) + \frac{i}{b+k}$ for $0 < \mathrm{Im}(k) < a$.

- $g_+(x)$ bdd as $x \to \infty \Rightarrow \bar{g}_+ \in H(\mathrm{Im}(k) > 0)$.

Thus, $-(k^2+a^2)^{1/2}A(k) = \bar{u}_y(k,0) = \bar{g}_+(k)$ for $\mathrm{Im}(k)>0$

Eliminate $A(k)$

$$\Rightarrow \quad \bar{g}_+(k) + (k^2+a^2)^{1/2}\left(\bar{f}_-(k) + \frac{i}{b+k}\right) = 0 \quad \text{for } 0 < \mathrm{Im}(k) < a$$

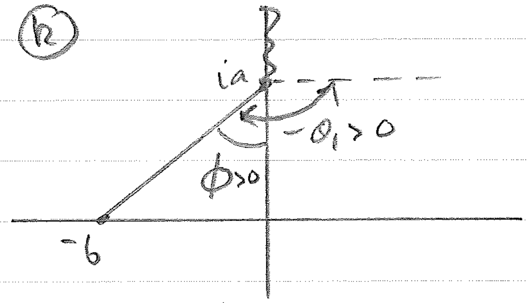
(ii) Split $(k^2+a^2)^{1/2}$, with branches of $(k \pm ia)^{1/2}$ as defined above $\Rightarrow$

$$\frac{\bar{g}_+(k)}{(k+ia)^{1/2}} + (k-ia)^{1/2}\bar{f}_-(k) = -i\,\frac{(k-ia)^{1/2}}{k+b} \quad \text{for } 0 < \mathrm{Im}(k) < a$$

Split RHS $\Rightarrow$

$$\frac{(k-ia)^{1/2}}{k+b} = \frac{(k-ia)^{1/2} - (-b-ia)^{1/2}}{k+b} + \frac{(-b-ia)^{1/2}}{k+b}$$

where $(-b-ia)^{1/2} = |-b-ia|^{1/2} e^{i\theta_1/2}$ and $\theta_1 = -(\pi/2+\phi)$, $\tan\phi = b/a$, with ϕ as shown



Hence, for $0 < \mathrm{Im}(k) < a$,

$$\frac{\bar{g}_+(k)}{(k+ia)^{1/2}} + \frac{i(-b-ia)^{1/2}}{k+b} = -(k-ia)^{1/2}\bar{f}_-(k) - \frac{i\left[(k-ia)^{1/2} - (-b-ia)^{1/2}\right]}{k+b}$$

LHS $\in H(\mathrm{Im}(k) > 0)$, RHS $\in H(\mathrm{Im}(k) < a)$ and LHS = RHS in overlap strip $0 < \mathrm{Im}(k) < a$
$\Rightarrow$ LHS is the analytic continuation of the RHS into the upper-half plane, so together they define an entire function, $E(k)$ say.

Since $k^{1/2}\bar{g}_+(k)$ and $k\bar{f}_-(k)$ are bounded at ∞, $E(k) \to 0$ as $|k| \to \infty$, so $E(k) \equiv 0$ by Liouville's theorem.

Hence, $$\bar{g}_+(k) = -\frac{i(-b-ia)^{1/2}(k+ia)^{1/2}}{k+b}$$

$$\bar{f}_-(k) = -\frac{i\left[(k-ia)^{1/2} - (-b-ia)^{1/2}\right]}{(k+b)(k-ia)^{1/2}}$$ $\square$

3(b) (i) $$\bar{h}_+(k) = \int_0^\infty e^{x(c+ik)} + e^{x(-1+ik)}\,dx$$

$$= \left.\frac{e^{x(c+ik)}}{c+ik} + \frac{e^{x(-1+ik)}}{-1+ik}\right|_0^\infty$$

$$= -\frac{1}{c+ik} - \frac{1}{-1+ik}$$

$$= \frac{i}{k-ic} + \frac{i}{k+i}$$

provided $\mathrm{Re}(c+ik) < 0$ and $\mathrm{Re}(-1+ik) < 0$, i.e. $\mathrm{Im}(k) > c$. $(\because c > 0)$

$$\bar{h}_-(k) = \int_{-\infty}^0 e^{x(-c+ik)} + e^{x(1+ik)}\,dx$$

$$= \left.\frac{e^{x(-c+ik)}}{-c+ik} + \frac{e^{x(1+ik)}}{+1+ik}\right|_{-\infty}^0$$

$$= +\frac{1}{-c+ik} + \frac{1}{1+ik}$$

$$= -\frac{i}{k+ic} - \frac{i}{k-i}$$

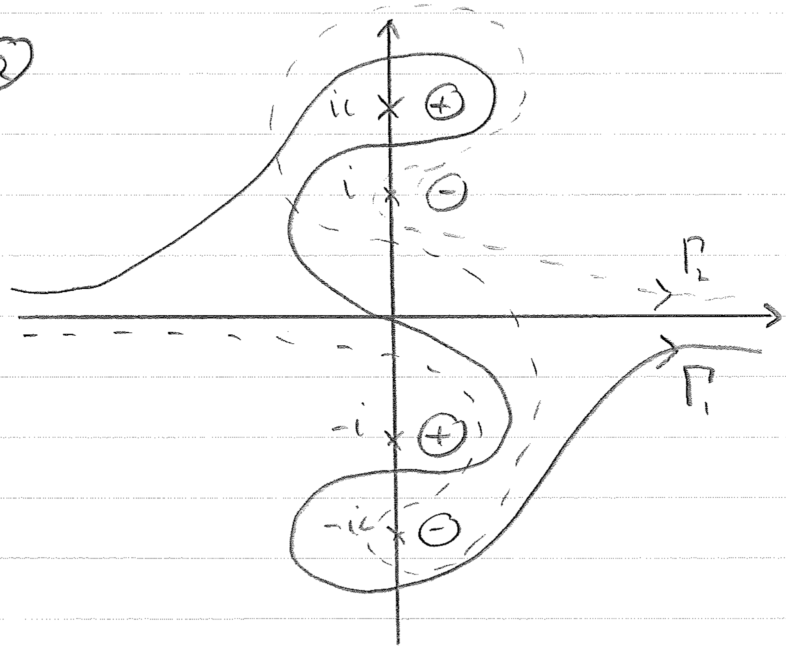
provided $\mathrm{Re}(-c+ik) > 0$ and $\mathrm{Re}(1+ik) > 0$, i.e. $\mathrm{Im}(k) < -c$

(ii) $\bar{h}_+ \in H(\mathbb{C}\setminus\{-i, ic\})$, $\bar{h}_- \in H(\mathbb{C}\setminus\{i, -ic\})$

$$\bar{h} := \bar{h}_- + \bar{h}_+ \in H(\mathbb{C}\setminus\{\pm i, \pm ic\})$$

(iii) Γ must pass above (below) singularities of $\bar{h}_+$ ($\bar{h}_-$), else $h_\pm(x) \neq 0$ for $\pm x < 0$.

$\boxed{c > 1}$ R



$\boxed{0<c<1}$ R