2B68

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C6.3b

APPLIED COMPLEX VARIABLES

TRINITY TERM 2014

FRIDAY, 6 JUNE 2014, 9.30am to 11.00am

You may submit answers to as many questions as you wish but only the best two will count for the total mark.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

Do not turn this page until you are told that you may do so

1. (a) [10 marks] Let Ω be a domain in the ζ-plane bounded by a polygon $\partial\Omega$ with exterior angles $\beta_j\pi$, $j = 1, \ldots, n$. The conformal map $\zeta = G(Z)$ maps the upper half-plane $\mathrm{Im}(Z) > 0$ onto Ω, with the finite points $X_1 < X_2 < \cdots < X_n$ on the real axis being mapped to the vertices $\zeta_j = G(X_j)$, $j = 1, \ldots, n$.

   (i) State and verify the Schwarz–Christoffel formula for $\mathrm{d}G/\,\mathrm{d}Z$. In general, how many of the X_j can be specified independently? How is the formula modified if $X_n = \infty$?

   (ii) By mapping to the Z-plane and seeking a solution that is a linear combination of $\arg(Z - X_1)$ and $\arg(Z - X_2)$, find a bounded function $T(\xi, \eta)$ that is harmonic for $\xi + \mathrm{i}\eta \in \Omega$ with $T = 0$ for $\xi + \mathrm{i}\eta \in \partial\Omega\backslash C$ and $T = 1$ for $\xi + \mathrm{i}\eta \in C$, where C is the edge of $\partial\Omega$ joining ζ_1 to ζ_2.

(b) [15 marks] Consider steady potential flow of a jet around a rigid wall $ABDE$ with a free surface $A'E'$, as illustrated below. The points A', A, B, D, E and E' lie at $(x, y) = (-2, -\infty)$, $(-1, -\infty)$, $(-1, 0)$, $(1, 0)$, $(1, -\infty)$ and $(2, -\infty)$, respectively, while C and F lie on $x = 0$. The walls AB, BD and DE are straight lines. The fluid velocity (u, v) tends to $(0, 1)$ at $A'A$ and to $(0, -1)$ at EE'. Take the stream function $\psi = 0$ on $ABDE$, so that $\psi = 1$ and $u^2 + v^2 = 1$ on $A'E'$. Take the potential $\phi = 0$ at C, so that $\phi = 0$ on $x = 0$. Take $\phi = -\phi^*$ at B, so that $\phi = \phi^*$ at D, where $\phi^* > 0$. Take $(u, v) = (u^*, 0)$ at C, where $u^* > 1$.

   (i) Sketch the potential plane $w = \phi + \mathrm{i}\psi$ and its image in the Z-plane under the map

$$Z = \frac{1 + \mathrm{e}^{\pi w}}{1 - \mathrm{e}^{\pi w}}.$$

   Sketch the hodograph plane $w' = u - \mathrm{i}v$ and its image in the ζ-plane under the map

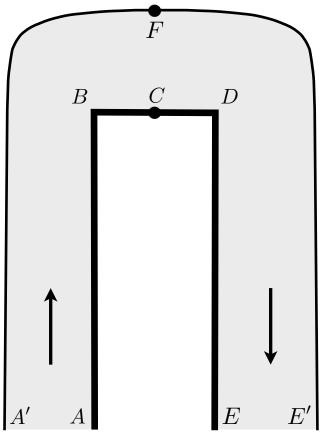
$$\zeta = \log(w'),$$

   where log denotes the principal branch of the logarithm. In your sketches label the locations of the points A', A, B, C, D, E, E' and F

   (ii) Deduce that

$$\zeta = Q \int_0^Z \frac{\mathrm{d}t}{(t^2 - 1)^{1/2}(t^2 - \coth^2(\pi\phi^*/2))}$$

   and derive two expressions relating the real constants Q, ϕ^* and u^*.



2. Let $\Gamma = \{x + \mathrm{i}y : |x| < 1,\, y = 0\}$ and $\overline{\Gamma} = \{x + \mathrm{i}y : |x| \leqslant 1,\, y = 0\}$. Suppose w is holomorphic away from $\overline{\Gamma}$ and $w_+ + w_- = g$ on Γ, where $w_\pm(x) = \lim_{y\to\pm 0} w(x + \mathrm{i}y)$ and g is a prescribed function that is holomorphic in an open set containing Γ and continuous on $\overline{\Gamma}$. Suppose $\widetilde{w}$ is holomorphic and non-zero away from $\overline{\Gamma}$ and that $\widetilde{w}_+ = -\widetilde{w}_- \neq 0$ on Γ.

(a) [12 marks] (i) Derive the Plemelj formulae on Γ for the Cauchy integral

$$I(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{F(\zeta)\,\mathrm{d}\zeta}{\zeta - z}$$

in which the density F is holomorphic in an open set containing Γ and continuous on $\overline{\Gamma}$, defining precisely the principal value integral $⨍$.

(ii) Determine the density F for which a solution for w is given by

$$\frac{w(z)}{\widetilde{w}(z)} = I(z) + H(z),$$

where H is an arbitrary function of z that is holomorphic away from $z = \pm 1$.

(iii) Define a branch of $(z^2 - 1)^{1/2}$ for which a solution for $\widetilde{w}$ is given by

$$\widetilde{w}(z) = \frac{1}{(z^2 - 1)^{1/2}}.$$

Use this solution for $\widetilde{w}(z)$ in parts (b) and (c).

(b) [8 marks] Suppose $w(z) = \mathrm{O}\left((z + 1)^m\right)$ as $z \to -1$ with $z \notin \Gamma$, $w(z) = \mathrm{O}\left((z - 1)^n\right)$ as $z \to 1$ with $z \notin \Gamma$ and $w(z) = \mathrm{O}\left(z^N\right)$ as $|z| \to \infty$. Find $H(z)$ in each of the following cases:

(i) $m = -1/2$, $n = -1/2$ and $N = -2$;

(ii) $m = -1/2$, $n = 1/2$ and $N = -1$;

(iii) $m = 1/2$, $n = 1/2$ and $N = 0$.

Is it possible to find a solution for w of the form in part (a)(ii) in the case in which $m = 1/2$, $n = 1/2$ and $N = -1$? Justify your answer.

[*You may assume that* $I(z) = O(1/z)$ *as* $|z| \to \infty$ *and that, for* $a = \pm 1$, *if* $F(a) = 0$, *then* $I(z) = I(a) + O(z - a)$ *as* $z \to a$ *with* $z \notin \Gamma$.]

(c) [5 marks] By writing w as a Cauchy integral and taking $H = 0$, or otherwise, show that

$$g(x) = -\frac{1}{\pi^2} ⨍_{-1}^{1} \frac{\widetilde{w}_+(\eta)}{\eta - x} \left(⨍_{-1}^{1} \frac{g(\xi)}{\widetilde{w}_+(\xi)(\xi - \eta)} \mathrm{d}\xi \right) \mathrm{d}\eta \quad \text{for} \quad |x| < 1.$$

3. Suppose that f is a continuous solution of the integral equation

$$f(x) = \int_0^\infty K(x-t)f(t)\,\mathrm{d}t + g(x) \quad \text{for} \quad -\infty < x < \infty, \tag{1}$$

where the kernel is given by

$$K(x) = \frac{(a^2-b^2)}{2a}\mathrm{e}^{-a|x|},$$

the prescribed continuous function $g(x)$ is of $\mathrm{O}(\mathrm{e}^{-c|x|})$ as $x \to \pm\infty$, and a, b and c are positive constants, with $a \neq b$.

(a) [6 marks] (i) Determine the Fourier transform $\overline{K}(k)$, stating clearly where the Fourier integral converges. To which parts of the complex k-plane may $\overline{K}(k)$ be analytically continued?

(ii) Where is $\overline{g}(k)$ holomorphic?

(iii) Suppose $f(x) = \mathrm{O}(\mathrm{e}^{-d|x|})$ as $x \to \pm\infty$ for some constant $d > 0$ and define

$$f_+(x) = \begin{cases} 0 & \text{for } x < 0, \\ f(x) & \text{for } x \geqslant 0, \end{cases} \qquad f_-(x) = \begin{cases} f(x) & \text{for } x < 0, \\ 0 & \text{for } x \geqslant 0. \end{cases}$$

Where are $\overline{f}_+(k)$ and $\overline{f}_-(k)$ holomorphic?

(b) [9 marks] (i) Show how to convert the integral equation (1) to

$$f_+(x) + f_-(x) = \int_{-\infty}^\infty K(x-t)f_+(t)\,\mathrm{d}t + g(x) \quad \text{for} \quad -\infty < x < \infty.$$

(ii) Deduce that

$$\left(\frac{k+\mathrm{i}b}{k+\mathrm{i}a}\right)\overline{f}_+(k) + \left(\frac{k-\mathrm{i}a}{k-\mathrm{i}b}\right)\overline{f}_-(k) = \left(\frac{k-\mathrm{i}a}{k-\mathrm{i}b}\right)\overline{g}(k) \quad \text{for} \quad \alpha < \mathrm{Im}(k) < \beta, \tag{2}$$

where α and β should be defined.

(iii) Outline a procedure for solving (2) for $\overline{f}_+(k)$ and $\overline{f}_-(k)$.

(c) [10 marks] Now consider the case in which $a \neq 1$, $b \neq 1$ and $g(x) = \mathrm{e}^{-|x|}$.

(i) Deduce from (2) expressions for $\overline{f}_+(k)$ and $\overline{f}_-(k)$.

(ii) Hence find a solution f of the integral equation (1).

[*You may assume that* $\overline{f}_\pm(k)$ *are of* $\mathrm{O}(k^{-1})$ *as* $|k| \to \infty$.]

Applied Complex Variables

C6.3b 2014 Q1

(a)(i) The tangent to $\partial\Omega$ has direction $\arg G'(Z)$ because $dZ = dX$ is real on $\partial\Omega$. Thus we require $\arg G'(Z)$ to be constant on each side of $\partial\Omega$ with

$$\left[\arg G'(Z)\right]_{X_j^-}^{X_j^+} = \beta_j \pi,$$

and in addition that $G'(Z) \neq 0$ for $Z \neq X_j$.

We see that

$$G'(Z) = Q \prod_{j=1}^{n} (Z - X_j)^{-\beta_j} \qquad (Q \in \mathbb{C})$$

has exactly these properties, because

$$\arg G'(Z) = \arg Q - \sum_{j=1}^{n} \beta_j \arg(Z - X_j),$$

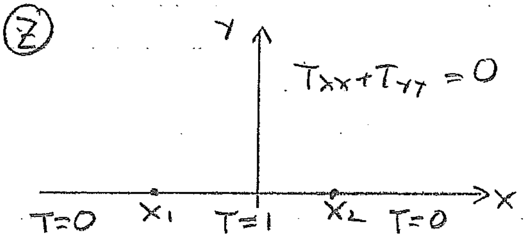
provided we take e.g. the principal branch of $(Z - X_j)^{-\beta_j}$.

In general three of the X_j can be specified independently, by the Riemann Mapping Theorem.

If $X_n = \infty$, the formula is modified to

(B6) $$G'(Z) = Q \prod_{j=1}^{n-1} (Z - X_j)^{-\beta_j}$$

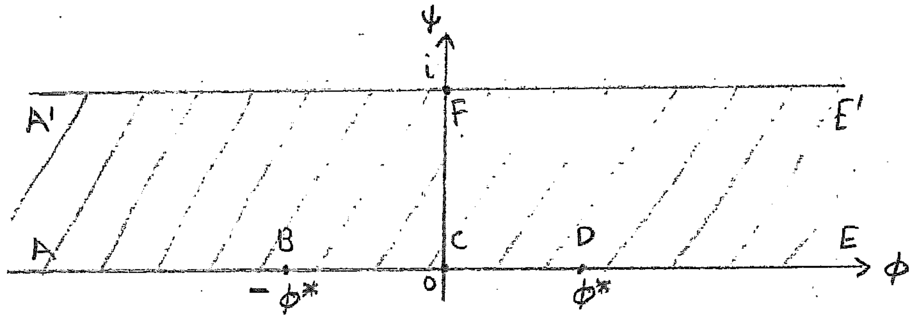
(a)(ii) Map to Z-plane $\Rightarrow$ (Z)



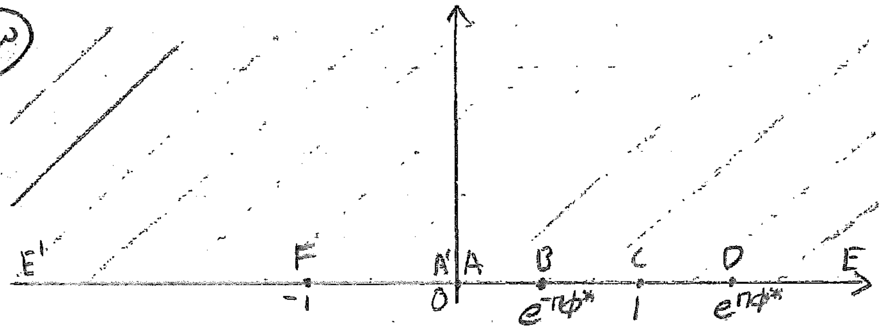
Bounded solution is $T = \frac{1}{\pi}\left(\arg(Z - X_2) - \arg(Z - X_1)\right)$, with principal branch

(S4) $\Rightarrow T = \frac{1}{\pi} \operatorname{Im}\left(\log\left(\frac{G^{-1}(z) - X_2}{G^{-1}(z) - X_1}\right)\right)$, with principal branch.

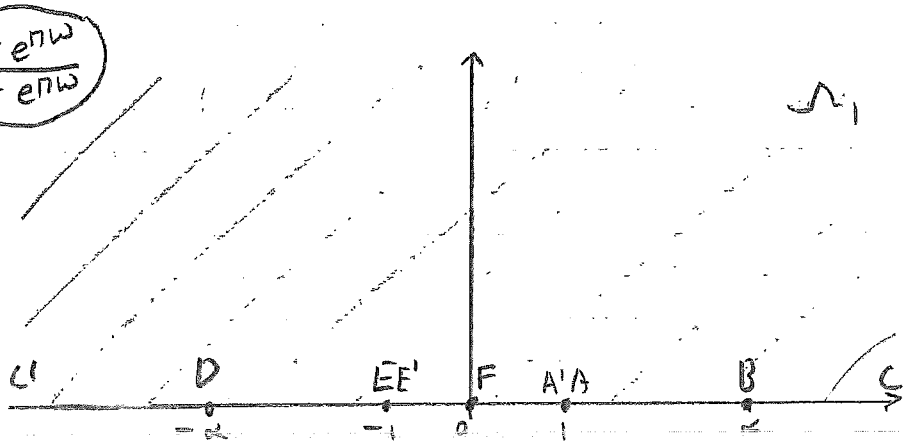
(b)(i) w



$e^{\pi w}$

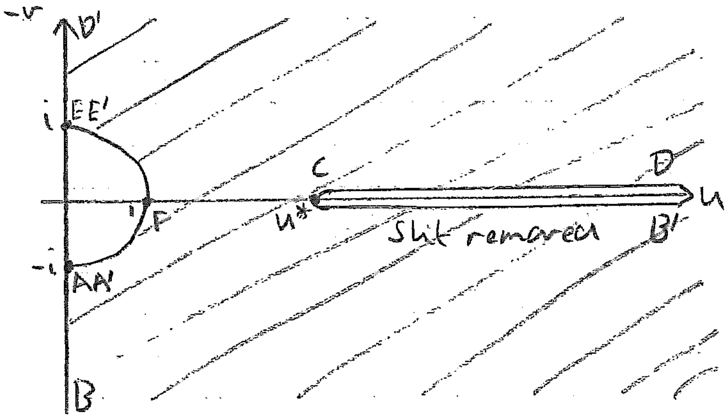


$z = \frac{1+e^{\pi w}}{1-e^{\pi w}}$

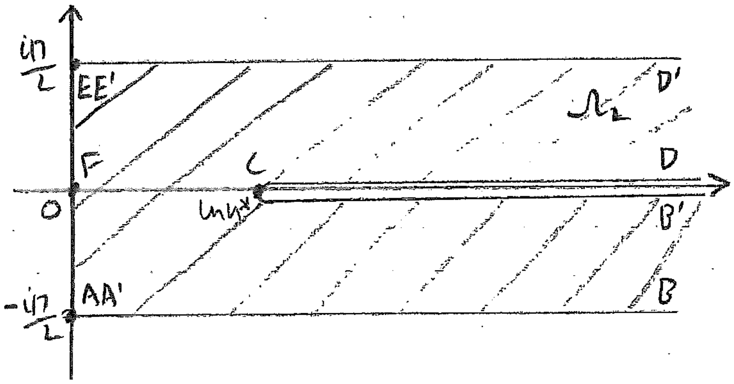


$(\alpha = \coth \frac{\pi\phi^*}{2})$

w'



$\zeta = \log w'$



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(b)(ii) Map A, C, E in Z-plane to A, C, E in ζ-plane using (a) with $n=5$, $\beta_A = \frac{1}{2}$, $\beta_B = 1$, $\beta_C = -1$, $\beta_D = 1$, $\beta_E = -\frac{1}{2}$ and $X_E = -1$, $X_A = 1$, $X_C = \infty$ $\Rightarrow$

$$\frac{dG}{dZ} = Q(Z-1)^{-1/2}(Z-X_B)^{-1}(Z-X_D)^{-1}(Z+1)^{-1/2},$$

Symmetry $\Rightarrow$ $G(0) = 0$ and if $X_B = \alpha$, then $X_D = -\alpha$

$$\Rightarrow \quad G(Z) = Q\int_0^Z \frac{dt}{(t^2-1)^{1/2}(t^2-\alpha^2)} \qquad \square$$

Take branch of $(Z^2-1)^{1/2}$ with cut $[-1,1]$ in Z-plane, say, that is real and positive for $\mathrm{Re}(Z) > 1$, $\mathrm{Im}(Z) = 0$.

$\mathrm{Re}(\zeta) = 0$ on AE $\Rightarrow$ $Q \in \mathbb{R}$

$$G(\pm 1) = \mp \frac{i\pi}{2} \text{ at A, E} \underset{\substack{\uparrow \\ \text{symmetry}}}{\Rightarrow} -\frac{i\pi}{2} = Q\int_0^1 \frac{dt}{+i(1-t^2)^{1/2}(t^2-\alpha^2)}$$

$$\Rightarrow \frac{\pi}{2} = Q\int_0^1 \frac{dt}{(1-t^2)^{1/2}(t^2-\alpha^2)} \qquad ①$$

$$\underset{\substack{\uparrow \\ \text{say}}}{G(i\infty)} = \ln u^* \text{ at C} \Rightarrow \ln u^* = Q\int_0^\infty \frac{i\,ds}{i(1+s^2)^{1/2}(-s^2-\alpha^2)} \qquad \begin{matrix}(t=is)\\ \text{wlog}\end{matrix}$$

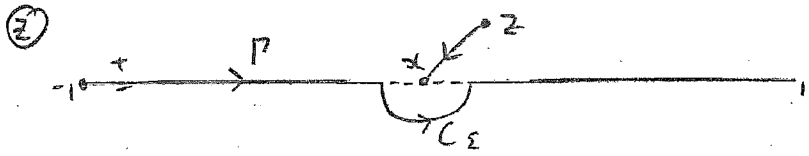
$$\Rightarrow \ln u^* = -Q\int_0^\infty \frac{ds}{(1+s^2)^{1/2}(\alpha^2+s^2)} \qquad ②$$

① & ② are two expressions relating Q, ϕ^* and u^* (via $\alpha = \coth(\frac{\pi\phi^*}{2})$)

N7

C6.3b 2014 Q2

(a)(i) As $z \to x \in \Gamma$ from the + side indent Γ with a small semi-circle C_ε around x as shown, where the radius ε is sufficiently small that F is holomorphic in $D(x, 2\varepsilon) = \{z : |z - x| < 2\varepsilon\}$, say.



Let $\gamma_\varepsilon = \Gamma \cap D(x; \varepsilon) \subseteq \Gamma$, i.e. the portion of Γ replaced by C_ε.

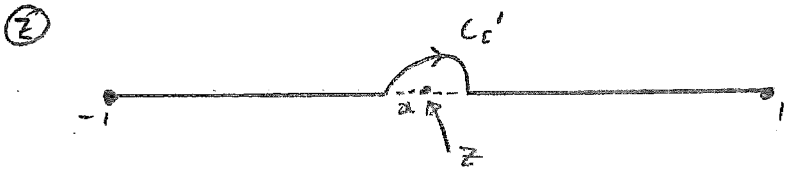
Deformation Thm $\Rightarrow$ $I(z) = \frac{1}{2\pi i}\left(\int_{\Gamma\setminus\gamma_\varepsilon} + \int_{C_\varepsilon}\right)\frac{F(\zeta)}{\zeta - z}\,d\zeta$

$$\underset{(z\to x)}{\Rightarrow} I_+(x) = \frac{1}{2\pi i}\left(\int_{\Gamma\setminus\gamma_\varepsilon} + \int_{C_\varepsilon}\right)\frac{F(\zeta)}{\zeta - x}\,d\zeta$$

$$\underset{(\varepsilon\to 0+)}{\Rightarrow} I_+(x) = \underbrace{\frac{1}{2\pi i}\fint_{\Gamma}\frac{F(\zeta)\,d\zeta}{\zeta - x}}_{\fint_\Gamma = \lim_{\varepsilon\to 0}\int_{\Gamma\setminus\gamma_\varepsilon}} + \underbrace{\frac{1}{2}\cdot\frac{2\pi i}{2\pi i}\cdot \mathrm{Res}_{\zeta = x}\frac{F(\zeta)}{\zeta - x}}_{=\frac{1}{2}F(x)}$$

The PVI exists ∵ F cts on $\bar{\Gamma}$ (so, in particular, log singularities cancel as $\varepsilon \to 0+$).

For $I_-(x)$ replace γ_ε with C_ε' as shown:



Argument same as above except semi-circle C_ε' now gives a contribution $-\frac{1}{2}\cdot\frac{2\pi i}{2\pi i}\cdot\mathrm{Res}_{\zeta = x}\frac{F(\zeta)}{\zeta - x} = -\frac{1}{2}F(x)$.

Hence, we obtain the Plemelj formulae:

$$I_\pm(x) = \pm\frac{1}{2}F(x) + \frac{1}{2\pi i}\fint_{-1}^{1}\frac{F(\zeta)}{\zeta - x}\,d\zeta \quad \text{for } x \in \Gamma.$$

(a)(ii) Let $W(z) = \dfrac{w(z)}{\tilde{w}(z)} = I(z) + H(z)$

(a)(i) $\Rightarrow$ $W_\pm = \pm\frac{1}{2}F + G + H_\pm$ on Γ,

where $G(x) = \dfrac{1}{2\pi i} \displaystyle⨍_{-1}^{1} \dfrac{F(\xi)}{\xi - x}\, d\xi$ for $x \in \Gamma$.

Subtract $\Rightarrow$

$$\begin{aligned} F &= W_+ - W_- \\ &= \frac{w_+}{\tilde{w}_+} - \frac{w_-}{\tilde{w}_-} \\ &= \frac{w_+ + w_-}{\tilde{w}_+} \\ &= \frac{g}{\tilde{w}_+} \quad \text{on } \Gamma \end{aligned}$$

(iii) Define e.g. $(z^2-1)^{1/2} = |z^2-1|^{1/2} \exp\big(i(\arg(z-1) + \arg(z+1))/2\big)$, where $\arg(z\pm1) \in (-\pi, \pi]$, so that branch cut coincides exactly with Γ.

B12

(b)(i) With $\tilde{w}(z) = \dfrac{1}{(z^2-1)^{1/2}}$ as in (a)(iii), $\tilde{w}_+(x) = \dfrac{1}{i(1-x^2)^{1/2}}$ for $x \in \Gamma$

$$\Rightarrow (z^2-1)^{1/2} w(z) = \frac{w(z)}{\tilde{w}(z)} = \underbrace{\frac{1}{2\pi i} \int_{-1}^{1} \frac{i(1-\xi^2)^{1/2} g(\xi)\, d\xi}{\xi - z}}_{I(z)} + \underbrace{H(z)}_{\in H(\mathbb{C}\setminus\{\pm1\})}$$

By hint, $I(z) = I(\pm1) + O(z\mp1)$ as $z \to \pm1$ with $z \notin \Gamma$ and

$$I(z) = O(1/z) \quad \text{as } |z| \to \infty.$$

Hence, in (i)-(iii), $H(z) = O(1)$ as $z \to \pm1$, so by Laurent's Thm H has removeable singularities at $z = \pm1$, and is therefore entire. It follows that

$$w(z) = \frac{I(\pm 1) + H(\pm 1)}{(z^2-1)^{1/2}} + O((z \mp 1)^{1/2}) \text{ as } z \to \pm 1. \qquad (+)$$

(i) $w(z) = O(z^{-2})$ as $|z| \to \infty \Leftrightarrow H(z) = O(z^{-1})$ as $|z| \to \infty$

Hence, by Liouville's Thm, $\underline{\underline{H \equiv 0}}$, and by $(+)$ we automatically have correct behaviour at $z = \pm 1$.

(ii) $w(z) = O(z^{-1})$ as $|z| \to \infty \Leftrightarrow H(z) \to$ constant as $|z| \to \infty$.

Hence, by Liouville's Thm, $H \equiv$ constant.

$w(z) = O((z+1)^{-1/2})$ as $z \to -1$ with $z \notin \Gamma$ automatically satisfied

$w(z) = O((z-1)^{+1/2})$ as $z \to +1$ with $z \in \Gamma \underset{(+)}{\Leftrightarrow} I(1) + H(1) = 0$

ⓈⒺ

$$\text{i.e. } \underline{\underline{H = -I(1) = \frac{1}{2\pi}\int_{-1}^{1}\left(\frac{1+\zeta}{1-\zeta}\right)^{1/2} g(\zeta)\,d\zeta}}.$$

(iii) $w(z) = O(1)$ as $|z| \to \infty \Leftrightarrow H(z) = O(z)$ as $|z| \to \infty$

Hence, by Liouville's Thm, $H = a + bz$ $(a, b \in \mathbb{C})$.

Now end point conditions require $I(\pm 1) + H(\pm 1) = 0$

$\Rightarrow I(\pm 1) + a \pm b = 0 \Rightarrow a = -\frac{1}{2}(I(1) + I(-1)),\ b = -\frac{1}{2}(I(1) - I(-1))$

$$\Rightarrow \underline{\underline{H = \frac{1}{2\pi}\int_{-1}^{1}\frac{\zeta g(\zeta)}{\sqrt{1-\zeta^2}}\,d\zeta + \frac{z}{2\pi}\int_{-1}^{1}\frac{g(\zeta)}{\sqrt{1-\zeta^2}}\,d\zeta}}$$

(iii) $\Rightarrow$ given solution exists for $m = n = 1/2$, $N = -1$

iff $b = 0$, i.e. $\displaystyle\int_{-1}^{1}\frac{g(\zeta)}{\sqrt{1-\zeta^2}}\,d\zeta = 0$.

(c) $\displaystyle w(z) = \frac{1}{2\pi i}\int_{-1}^{1}\frac{f(\zeta)\,d\zeta}{\zeta - z} \underset{(a)(ii)}{\Rightarrow} \underset{\text{on } \Gamma}{f = w_+ - w_-},\ \frac{1}{\pi i}-\!\!\!\!\!\!\int_{-1}^{1}\frac{f(\zeta)\,d\zeta}{\zeta - x} = w_+(x) + w_-(x)$ for $x \in \Gamma$

But with $H = 0$, $w_+ - w_- = \tilde{\omega}_+(W_+ + W_-) = 2\tilde{\omega}_+ G = f$

$w_+ + w_- = \tilde{\omega}_+(W_+ - W_-) = \tilde{\omega}_+ F = g$, both on Γ

Hence, on Γ, $\displaystyle g(x) = \frac{1}{\pi i}-\!\!\!\!\!\!\int_{-1}^{1}\frac{2\tilde{\omega}_+(\eta)G(\eta)}{\eta - x}\,d\eta,\ G(\eta) = \frac{1}{2\pi i}-\!\!\!\!\!\!\int_{-1}^{1}\frac{g(\zeta)\,d\zeta}{\tilde{\omega}_+(\zeta)(\zeta - \eta)}$

and eliminating $G(\eta)$ gives the result.

Ⓢ5

C6.3b 2014 Q3

(a)(i) $$\bar{K}(k) = \frac{a^2-b^2}{2a}\left[\int_{-\infty}^{0} e^{(a+ik)x}\,dx + \int_0^{\infty} e^{(-a+ik)x}\,dx\right]$$

$$= \frac{a^2-b^2}{2a}\left[\left.\frac{e^{(a+ik)x}}{a+ik}\right|_{x=-\infty}^{x=0} + \left.\frac{e^{(-a+ik)x}}{-a+ik}\right|_{x=0}^{x=\infty}\right]$$

$$= \frac{a^2-b^2}{2a}\left[\frac{1}{a+ik} - \frac{1}{-a+ik}\right] \qquad (\operatorname{Re}(a+ik) > 0 \;\&\; \operatorname{Re}(-a+ik) < 0)$$

$$= \frac{a^2-b^2}{k^2+a^2} \qquad (|\operatorname{Im}(k)| < a)$$

which can be continued into $\mathbb{C} | \{-ia, ia\}$, i.e. $\bar{K} \in H(\mathbb{C}|\{\pm ia\})$

(ii) $g \in C(\mathbb{R})$ & $g = o(e^{-c|x|})$ as $x \to \pm\infty \Rightarrow \bar{g} \in H(|\operatorname{Im}(k)| < c)$.

(iii) $f \in C(\mathbb{R})$ & $f = o(e^{-d|x|})$ as $x \to \pm\infty \Rightarrow$,

$$\bar{f}_+(k) = \int_0^{\infty} f(x)e^{ikx}\,dx \in H(\operatorname{Im}(k) > -d),$$

$$\bar{f}_-(k) = \int_{-\infty}^{0} f(x)e^{-ikx}\,dx \in H(\operatorname{Im}(k) < +d).$$

B6

(b)(i) $f(x) = f_+(x) + f_-(x)$ and $\int_0^{\infty} K(x-t)f(t)\,dt = \int_{-\infty}^{\infty} K(x-t)f_+(t)\,dt$ for $-\infty < x < \infty$, so integral equation becomes

$$f_+(x) + f_-(x) = \int_{-\infty}^{\infty} K(x-t)f_+(t)\,dt + g(x) \quad \text{for } -\infty < x < \infty$$

(ii) Take the Fourier transform of this equation applying the convolution theorem to obtain

$$\bar{f}_+(k) + \bar{f}_-(k) = \bar{K}(k)\bar{f}_+(k) + \bar{g}(k)$$

wherever all of the Fourier transforms exist, i.e.

$$\{\operatorname{Im}(k) > -d\} \cap \{\operatorname{Im}(k) < d\} \cap \{\mathbb{C}|\{\pm ia\}\} \cap \{|\operatorname{Im}(k)| < c\}.$$

$= D$ say

Substitute for $\bar{K}(k) \Rightarrow$

$$\bar{f}_+(k) + \bar{f}_-(k) = \left(\frac{a^2-b^2}{k^2+a^2}\right)\bar{f}_+(k) + \bar{g}(k)$$

$$\Rightarrow \quad \left(\frac{k^2+b^2}{k^2+a^2}\right)\bar{f}_+(k) + \bar{f}_-(k) = \bar{g}(k)$$

$$\Rightarrow \quad \left(\frac{k+ib}{k+ia}\right)\bar{f}_+(k) + \left(\frac{k-ia}{k-ib}\right)\bar{f}_-(k) = \left(\frac{k-ia}{k-ib}\right)\bar{g}(k) \qquad (1)$$

for $k \in D\setminus\{ib\}$ and therefore for $\alpha < \mathrm{Im}(k) < \beta$

(S4) where $\max(-a,-c,-d) \leq \alpha < \beta \leq \min(a,b,c,d)$.

(iii) Can apply Wiener-Hopf method to (1):

① Find $G_- \in H(\mathrm{Im}(k) < \beta_1)$, $G_+ \in H(\mathrm{Im}(k) > \alpha_1)$ s.t.

RHS(1) $= G_-(k) + G_+(k)$ on $\Omega = \{\alpha_1 < \mathrm{Im}(k) < \beta_2\}$, with $\alpha \leq \alpha_1 < \beta_1 \leq \beta$.

Then (1) $\Rightarrow \left(\frac{k+ib}{k+ia}\right)\bar{f}_+(k) - G_+(k) = G_-(k) - \left(\frac{k-ia}{k-ib}\right)\bar{f}_-(k)$ on Ω (2)

② LHS(2) $\in H(\mathrm{Im}(k) > \alpha_1)$, RHS(2) $\in H(\mathrm{Im}(k) < \beta_1)$ and $\alpha_1 < \beta_1$, so LHS(2) = RHS(2) on a dense set of points, and therefore the RHS(2) is the (unique) analytic continuation of the LHS(2) into the lower half-plane, and together they define an entire function, $E(k)$ say.

③ Behaviour of $\bar{f}_\pm(k)$ and $G_\pm(k)$ as $|k| \to \infty$ $\Rightarrow$ behaviour of $E(k)$ as $|k| \to \infty$. If $E(k) = O(k^n)$ as $n \to \infty$ for some $n \in \mathbb{N}_0$, then Liouville's Thm $\Rightarrow E(k) = p_n(k)$, an arbitrary polynomial of degree n.

(B5) ④ Thus, (2) $\Rightarrow \bar{f}_+ = \left(\frac{k+ia}{k+ib}\right)(G_+ + p_n)$, $\bar{f}_- = \left(\frac{k-ib}{k-ia}\right)(G_- - p_n)$

(c)(i) (a)(i) $\Rightarrow \bar{g}(k) = \dfrac{2}{1+k^2}$ for $k \in \mathbb{C} \setminus \{\pm i\}$ & $c = 1.$

$$\Rightarrow \text{RHS}(1) = \frac{k-ia}{k-ib}\,\frac{2}{1+k^2}$$

$$= \frac{A}{k-i} + \frac{B}{k+i} + \frac{C}{k-ib}$$

where $A = -i\,\dfrac{a-1}{b-1}$, $B = i\,\dfrac{a+1}{b+1}$, $C = 2i\,\dfrac{a-b}{b^2-1}$.

Thus, (2) gives

$$\left(\frac{k+ib}{k+ia}\right)\bar{f}_+(k) - \frac{B}{k+i} = -\left(\frac{k-ia}{k-ib}\right)\bar{f}_- + \frac{A}{k-i} + \frac{C}{k-ib} \qquad (3)$$

for $\alpha < \text{Im}(k) < \beta$ as in (a)(ii), but with $c = 1$.

(a)(iii) ② $\Rightarrow$ $\text{LHS}(3) = \text{RHS}(3) \doteq E(k) \in H(\mathbb{C})$

$f_\pm(k) = O(k^{-1})$ as $|k| \to \infty$ $\Rightarrow$ $E(k) \to 0$ as $|k| \to \infty$

Liouville's Thm $\Rightarrow$ $E(k) = \text{constant} = 0$

(S6) Hence $\underline{\bar{f}_+(k) = B\dfrac{(k+ia)}{(k+i)(k+ib)}, \quad \bar{f}_-(k) = A\dfrac{(k-ib)}{(k-i)(k-ia)} + C\dfrac{1}{k-ia}.}$

(c)(ii) For $x > 0$, close at $-i\infty$ with inversion contour above singularities of $\bar{f}_+(k) \Rightarrow$

$$f(x) = -\frac{2\pi i}{2\pi}\left[\text{Res}_{k=-i} + \text{Res}_{k=-ib}\right]\bar{f}_+(k)e^{-ikx}$$

$$= -i\left[\frac{B(-i+ia)}{(-i+ib)}e^{-i(-i)x} + \frac{B(-ib)}{(-ib+i)}e^{-i(-ib)x}\right]$$

$$= \left(\frac{a^2-1}{b^2-1}\right)e^{-x} + \frac{(a+1)(b-a)}{b^2-1}e^{-bx}$$

For $x < 0$, close at $+i\infty$ with inversion contour below singularities of $\bar{f}_-(k) \Rightarrow$

$$f(x) = +\frac{2\pi i}{2\pi}\left[\text{Res}_{k=i} + \text{Res}_{k=ia}\right]\bar{f}_-(k)e^{-ikx}$$

$$= i\left[\frac{A(i-ib)}{(i-ia)}e^{-i(i)x} + A\frac{(ia-ib)}{(ia-i)}e^{-i(ia)x} + Ce^{-i(ia)x}\right]$$

(N4)

$$= \left(\frac{a-1}{b-1}\right)e^{x} - 2\left(\frac{a-b}{b^2-1}\right)e^{ax}.$$