A12207W1

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.6
Honour School of Mathematics and Statistics Part C: Paper C5.6

APPLIED COMPLEX VARIABLES

TRINITY TERM 2015

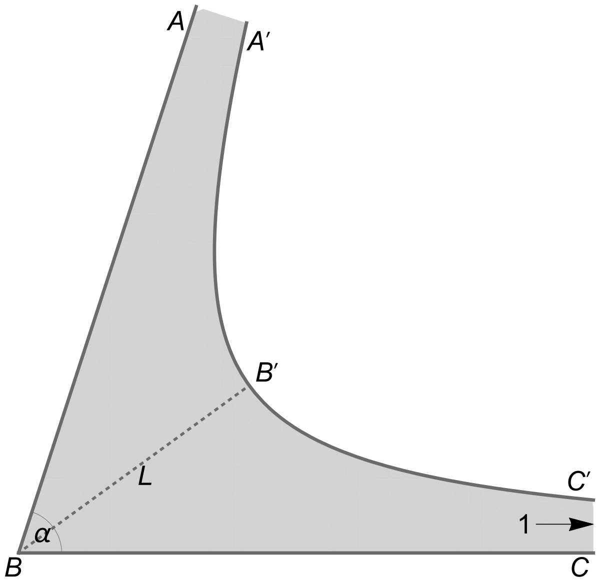
MONDAY, 8 JUNE 2015, 2.30pm to 4.00pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

Do not turn this page until you are told that you may do so

1. Consider the steady two-dimensional potential flow illustrated below. A jet of fluid flows between a free surface $A'B'C'$ and a rigid boundary ABC, consisting of two semi-infinite straight line segments which meet at an angle $\alpha < \pi$. The fluid layer has unit thickness and unit speed far downstream at CC'.



Axes are chosen such that A, B and C lie at $\infty(\cos\alpha, \sin\alpha)$, $(0, 0)$ and $(\infty, 0)$ respectively. The point labelled B' is at $L\big(\cos(\alpha/2), \sin(\alpha/2)\big)$, where L is the distance of closest approach of the free surface to the corner at B. The fluid velocity (u, v) is equal to $(1, 0)$ at CC' and $(-\cos\alpha, -\sin\alpha)$ at AA'. Bernoulli's Theorem implies that $u^2 + v^2 = 1$ on the free surface $A'B'C'$. The streamfunction ψ is taken to be equal to 0 on ABC and equal to 1 on $A'B'C'$. Finally, the potential ϕ is defined to be zero at the corner B.

(a) [7 marks] Sketch the potential plane ($w = \phi + \mathrm{i}\psi$) and the hodograph plane ($w' = u - \mathrm{i}v$). Explain briefly why the fluid domain is mapped onto respectively a strip and a sector of the unit disk, and label clearly the locations of the points A, B, C, A', B' and C' in each plane.

(b) [6 marks] By conformally mapping both the potential plane and the hodograph plane onto the quadrant $0 \leqslant \arg\zeta \leqslant \pi/2$, or otherwise, show that

$$\mathrm{e}^{\pi w/2} = \frac{1 + (w')^\beta}{1 - (w')^\beta}, \qquad \text{where} \quad \beta = \frac{\pi}{\pi - \alpha}.$$

(c) [7 marks] By setting $w' = \mathrm{e}^{\mathrm{i}\theta}$ on the free surface $A'B'C'$, show that the free surface is given parametrically by $z = z(\theta)$, where

$$\frac{\mathrm{d}z}{\mathrm{d}\theta} = -\frac{2\beta}{\pi}\frac{\mathrm{e}^{-\mathrm{i}\theta}}{\sin(\beta\theta)}.$$

State the boundary conditions for $z(\theta)$ and the range of θ.

(d) [5 marks] Show that the distance L from B to B' is given by

$$L = \sec\left(\frac{\pi}{2\beta}\right)\left(1 + \frac{2\beta}{\pi}\int_0^{\pi/2\beta} \frac{\sin(\theta)}{\sin(\beta\theta)}\,\mathrm{d}\theta\right).$$

2. (a) [6 marks] Let Γ be a smooth open contour in the complex plane and

$$w(z) = \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(\zeta)\,\mathrm{d}\zeta}{\zeta - z}, \qquad (\dagger)$$

where f is continuous on Γ and holomorphic in a neighbourhood of $t \in \Gamma$. Show that the limiting values of $w(z)$ as Γ is approached from either side are $w_\pm(t)$, where

$$w_\pm(t) = \pm\frac{1}{2} f(t) + \frac{1}{2\pi \mathrm{i}} ⨍_\Gamma \frac{f(\zeta)\mathrm{d}\zeta}{\zeta - t}$$

and you should define the integral $⨍$ precisely.

(b) [6 marks] Now let $\Gamma = \{x + \mathrm{i}y : |x| < c,\ y = 0\}$ and let $\overline{\Gamma} = \{x + \mathrm{i}y : |x| \leqslant c,\ y = 0\}$ be the closure of Γ, where $c > 0$ is a constant. Suppose that the function $\phi(x, y)$ satisfies Laplace's equation for $x + \mathrm{i}y \notin \overline{\Gamma}$, with $\phi(x, y) \to 0$ as $x^2 + y^2 \to \infty$ and $\partial\phi/\partial y = g(x)$ for $x + \mathrm{i}y \in \Gamma$, where g is holomorphic in a neighbourhood of Γ. Show that if $\partial\phi/\partial y = \mathrm{Im}\big[w(z)\big]$, where $w(z)$ is written in the form ($\dagger$), then f satisfies the singular integral equation

$$\frac{1}{2\pi} ⨍_{-c}^{c} \frac{f(\xi)\,\mathrm{d}\xi}{\xi - t} = -g(t)$$

for $t \in (-c, c)$.

(c) [6 marks] Clearly define a branch of the multifunction $\sqrt{z^2 - c^2}$ that is holomorphic on $\mathbb{C} \setminus \overline{\Gamma}$. Hence show that possible solutions for $w(z)$ take the form

$$w(z) = \frac{\mathrm{i}}{\pi\sqrt{z^2 - c^2}} \left(\int_{-c}^{c} \frac{g(\xi)\sqrt{c^2 - \xi^2}}{\xi - z}\,\mathrm{d}\xi + H(z) \right),$$

where $H(z)$ is holomorphic in $\mathbb{C} \setminus \{-c, c\}$ and real on Γ.

(d) [7 marks]

(i) If $w(z)$ is required to have at worst inverse square root singularities at $z = \pm c$, so that $\sqrt{z^2 - c^2}\, w(z)$ is bounded as $z \to \pm c$, and $zw(z) \to 0$ as $z \to \infty$, what can be said about $H(z)$?

(ii) Suppose instead that $w(z)$ is bounded as $z \to \pm c$ and that $w(z) = O(1/z)$ as $z \to \infty$. Show that such a solution exists only if g satisfies the condition

$$\int_{-c}^{c} \frac{g(\xi)\,\mathrm{d}\xi}{\sqrt{c^2 - \xi^2}} = 0.$$

3. (a) [6 marks] Let

$$K(x) = \begin{cases} e^{x} & x < 0, \\ e^{-3x} & x \geqslant 0, \end{cases} \qquad g_+(x) = \begin{cases} 0 & x < 0, \\ x & x \geqslant 0. \end{cases}$$

Compute the Fourier transforms $\overline{K}(k) = \int_{-\infty}^{\infty} K(x)e^{ikx}\,dx$ and $\overline{g}_+(k) = \int_{-\infty}^{\infty} g_+(x)e^{ikx}\,dx$. For what values of k are $\overline{K}(k)$ and $\overline{g}_+(k)$ defined? To what regions of the complex k-plane may each be analytically continued?

(b) [6 marks] Suppose that $f(x)$ satisfies the integral equation

$$\int_0^{\infty} K(x-t)f(t)\,dt = f(x) + x \quad \text{for } x \geqslant 0, \qquad (\star)$$

where K is as in part (a). Assume that $f(x) : [0,\infty) \to \mathbb{R}$ is continuous and that $f(x) = O(x)$ as $x \to \infty$.

Define

$$f_+(x) = \begin{cases} 0 & x < 0 \\ f(x) & x \geqslant 0 \end{cases}, \qquad h_-(x) = \begin{cases} \int_0^{\infty} K(x-t)f(t)\,dt & x < 0 \\ 0 & x \geqslant 0 \end{cases},$$

and show that the corresponding Fourier transforms satisfy the equation

$$\left(\overline{K}(k) - 1\right)\overline{f}_+(k) - \overline{g}_+(k) = \overline{h}_-(k),$$

where g_+ is as in part (a). State for which regions of the complex k-plane the Fourier transforms $\overline{f}_+(k)$ and $\overline{h}_-(k)$ are holomorphic.

(c) [5 marks] Show that

$$-\frac{(k+i)^2}{k+3i}\overline{f}_+(k) + \frac{k-i}{k^2} = (k-i)\overline{h}_-(k)$$

in a strip $\alpha < \mathrm{Im}(k) < \beta$, and give suitable values for α and β. Explain clearly why the left- and right-hand sides of this equation must be constant.

[*You may assume without proof that* $\overline{f}_+(k) = O(1/k)$ *as* $k \to \infty$.]

(d) [8 marks] Hence solve for $\overline{f}_+(k)$ and, by inverting the Fourier transform, show that the integral equation $(\star)$ is satisfied by

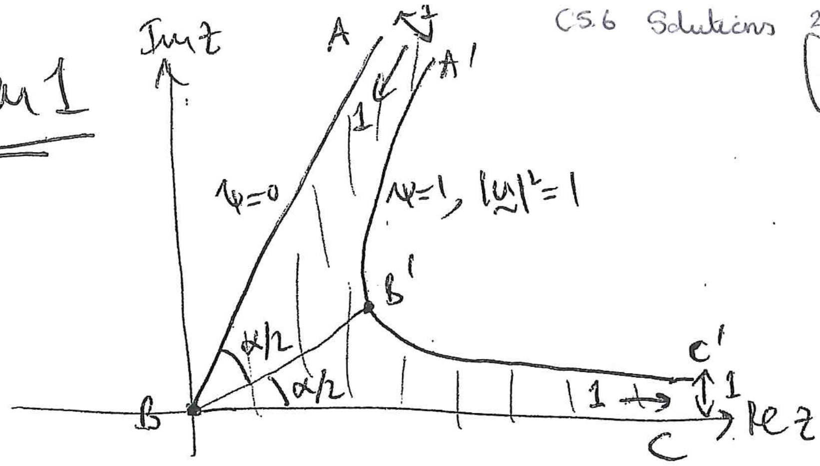
$$f(x) = 3x - 8 + e^{-x}\left[8 + 4x + C(1+2x)\right],$$

where C is an arbitrary constant.

[*You may assume without proof that there is no contribution from appropriately closing the inversion contour at infinity.*]

Question 1

[New application of familiar ideas]



(a) <u>Potential plane</u> $\psi = \operatorname{Im} w = 0$ on ABC

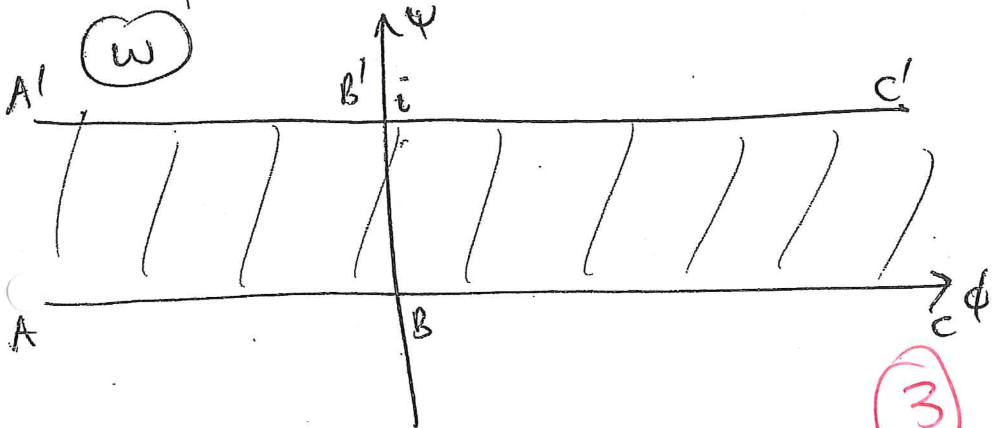
$\psi = 1$ on $A'B'C'$

at CC', $w \sim z \Rightarrow \phi \sim x \to +\infty$

at AA', $w \sim z e^{-i(\alpha+\pi)} = -z e^{-i\alpha}$

$\Rightarrow \phi \sim -x\cos\alpha - y\sin\alpha \to -\infty$

So potential plane looks like this:



- Given $w=0$ at B.
- By symmetry, velocity is orthogonal to BB'

$\Rightarrow \phi = \text{constant} = 0$ on BB'

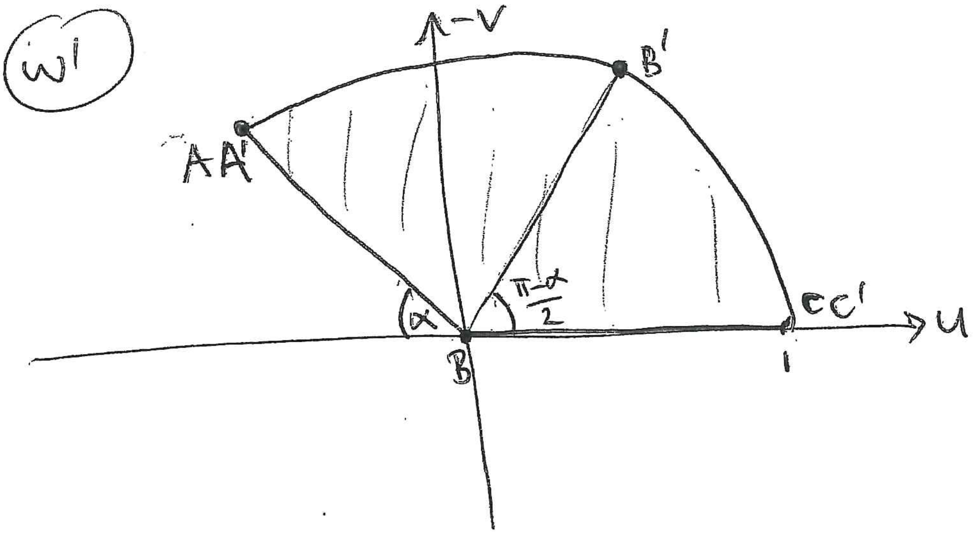
$\Rightarrow$ <u>B' is on $\phi=0$ line in w-plane</u>.

3

<u>Hodograph plane</u>

Given $\alpha < \pi$, we know that the velocity is zero at the corner B. So on BC we have $v=0$ and u increases from 0 (at B) to 1 (at C).

Similarly on AB, $(u,v) = V(-\cos\alpha, -\sin\alpha)$ where V decreases from 1 (at A) to zero (at B).

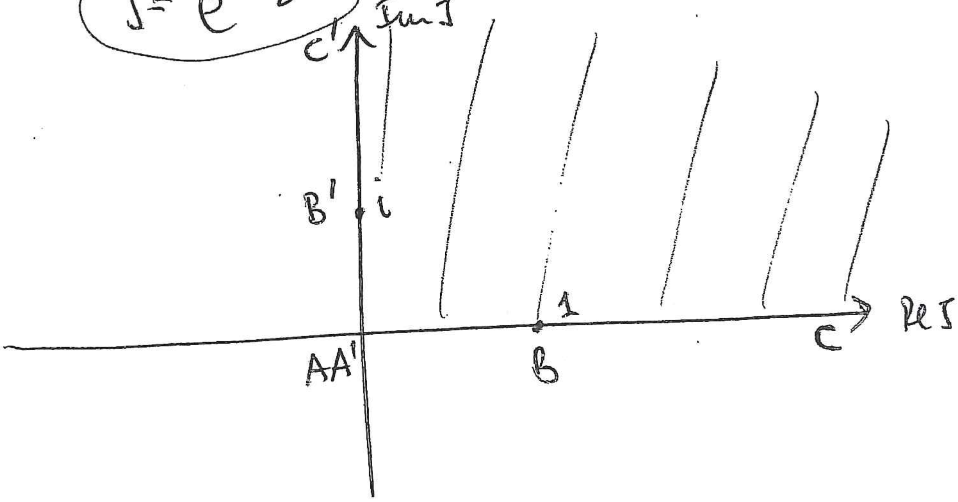


• on $A'B'C'$, $(u,v) = (\cos\theta, \sin\theta)$, where θ increases from $-\pi+\alpha$ (at A') to $\frac{-\pi+\alpha}{2}$ (at B') to zero (at C'

4 7

(b) Conformal mapping of w plane: [mappings used all standard]

$\zeta = e^{\frac{\pi w}{2}}$



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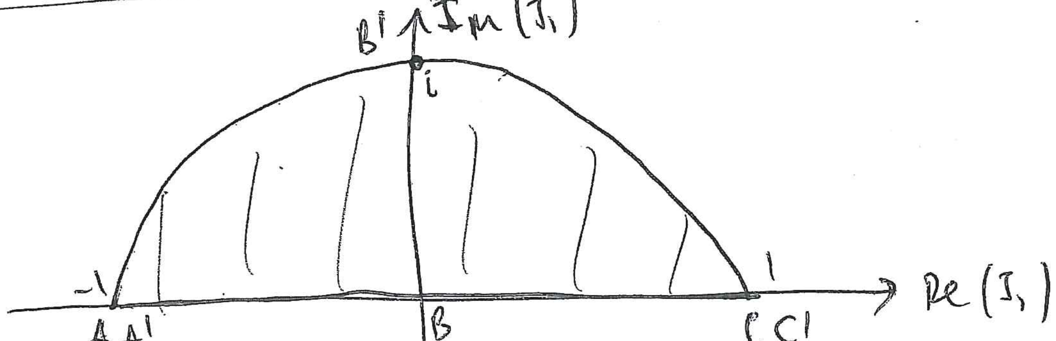
conformal mapping of w' plane:

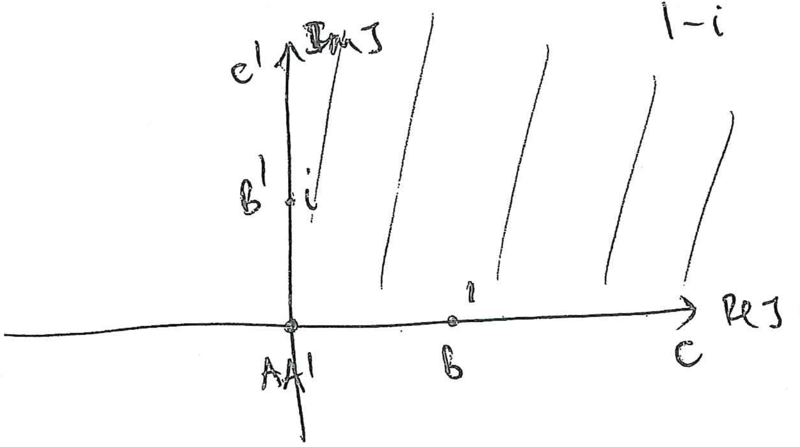
$\boxed{\zeta_1 = (w')^{\pi/(\pi-\alpha)} = (w')^{\beta}}$ maps to half-disk:



Now Möbius transformation taking -1 to 0, 0 to 1, 1 to ∞

$$\boxed{\zeta = \frac{1+\zeta_1}{1-\zeta_1}}$$. Check that $\zeta_1 = i$ maps to $\frac{1+i}{1-i} = \frac{(1+i)^2}{2} = i$

So



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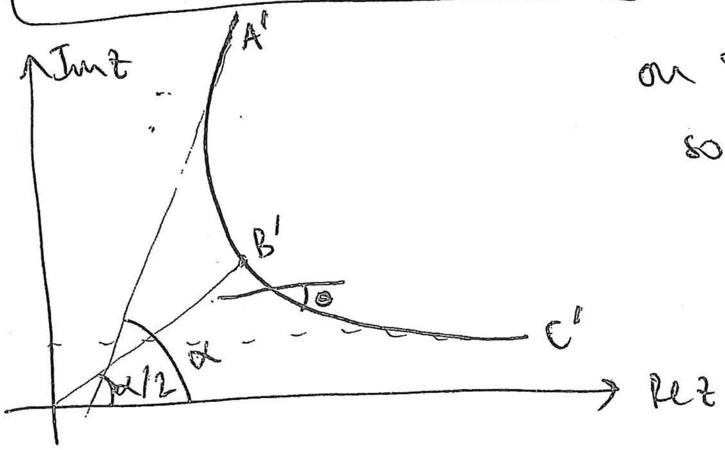
This is the same quadrant as for the w plane. Hence

$$\boxed{e^{\pi w/2} = \frac{1+(w')^{\beta}}{1-(w')^{\beta}}}$$.

1

6

(c)



on the surface, $|w'| = 1$
so $w' = u - iv = e^{i\theta}$ where
θ = angle shown.

$\theta \to 0$ at C'; then θ increases to $\frac{\pi-\alpha}{2} = \frac{\pi}{2\beta}$ at B',

then to $\pi - \alpha = \frac{\pi}{\beta}$ at A'

So $w' = e^{i\theta} \Rightarrow e^{\pi w/2} = \frac{1+e^{i\beta\theta}}{1-e^{i\beta\theta}} = i\cot\left(\frac{\beta\theta}{2}\right)$

Now $\frac{\pi}{2} e^{\frac{\pi w}{2}} w' \frac{dz}{d\theta} = -\frac{i}{2}\beta \operatorname{cosec}^2\left(\frac{\beta\theta}{2}\right)$

$$\therefore \frac{\pi}{2}\cdot\left(i\cot\left(\frac{\beta\theta}{2}\right)\right)\cdot e^{i\theta}\frac{dz}{d\theta} = -\frac{i}{2}\beta\operatorname{cosec}^2\left(\frac{\beta\theta}{2}\right)$$

$$\therefore \pi e^{i\theta}\frac{dz}{d\theta} = \frac{-\beta}{\cos\left(\frac{\beta\theta}{2}\right)\sin\left(\frac{\beta\theta}{2}\right)} = \frac{-2\beta}{\sin(\beta\theta)}$$

$$\therefore \boxed{\frac{dz}{d\theta} = -\frac{2\beta}{\pi}\frac{e^{-i\theta}}{\sin(\beta\theta)}}$$

[New application of familiar ideas] ~~4~~

For field conditions: $\boxed{\operatorname{Im} z \to 1 \text{ as } \theta \to 0}$

$$\boxed{\operatorname{Im}\left(z e^{\frac{i\pi}{\beta}}\right) \to 1 \text{ as } \theta \to \pi/\beta}$$ 3

[far field conditions are tricky!] ~~7~~

and $\underbrace{z = l e^{\frac{i\alpha}{2}} \text{ when } \theta = \frac{\pi}{2\beta}}$

where $\alpha = \pi - \frac{\pi}{\beta}$

$$\therefore \boxed{z = li\, e^{-i\pi/2\beta} \text{ at } \theta = \frac{\pi}{2\beta}}$$ 2

To get equation for l, just solve for y:

$$\frac{dy}{d\theta} = \frac{2\beta}{\pi}\frac{\sin\theta}{\sin(\beta\theta)} \quad \text{with } y(0) = 1,\ y\left(\frac{\pi}{2\beta}\right) = l\cos\left(\frac{\pi}{2\beta}\right)$$

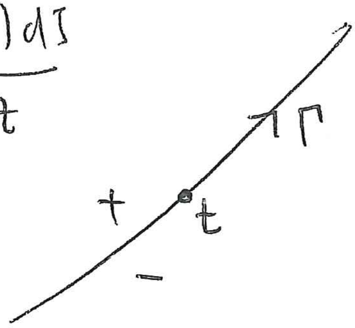
$$\frac{2\beta}{\pi}\int_0^{\frac{\pi}{2\beta}} \frac{\sin\theta}{\sin(\beta\theta)}\,d\theta = l\cos\left(\frac{\pi}{2\beta}\right) - 1$$ 3 5

$$\therefore \boxed{l = \sec\left(\frac{\pi}{2\beta}\right)\left[1 + \frac{2\beta}{\pi}\int_0^{\frac{\pi}{2\beta}} \frac{\sin\theta}{\sin(\beta\theta)}\,d\theta\right]}$$

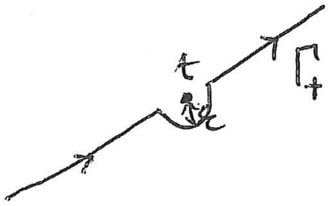
[New example]

Question 2

(a) $w(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - z}$



As $z \to t \in \Gamma$ from + side, deform contour as follows:



$$w_+(t) = \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_{\Gamma_+} \frac{f(\zeta)\,d\zeta}{\zeta - t}$$

$$= \lim_{\varepsilon \to 0} \left[\frac{1}{2\pi i} \int_{\Gamma \setminus D(t;\varepsilon)} \frac{f(\zeta)\,d\zeta}{\zeta - t} + \frac{1}{2\pi i} \int_{C_\varepsilon} \frac{f(\zeta)\,d\zeta}{\zeta - t} \right]$$

2nd integral $= \frac{1}{2} \times 2\pi i \times$ residue at $\zeta = t$

$$\therefore w_+(t) = \frac{1}{2\pi i} \fint_\Gamma \frac{f(\zeta)\,d\zeta}{\zeta - t} + \frac{1}{2} f(t) \quad 2$$

where $\fint_\Gamma \frac{f(\zeta)}{\zeta - t}\,d\zeta = \lim_{\varepsilon \to 0} \int_{\Gamma \setminus D(t;\varepsilon)} \frac{f(\zeta)\,d\zeta}{\zeta - t}$ 2

sim for w_-

$$w_-(t) = \frac{1}{2\pi i} \lim_{\varepsilon \to 0} \left[\int_{\Gamma \setminus D(t;\varepsilon)} \frac{f(\zeta)}{\zeta - t}\,d\zeta + \int_{C_\varepsilon} \frac{f(\zeta)}{\zeta - t}\,d\zeta \right]$$

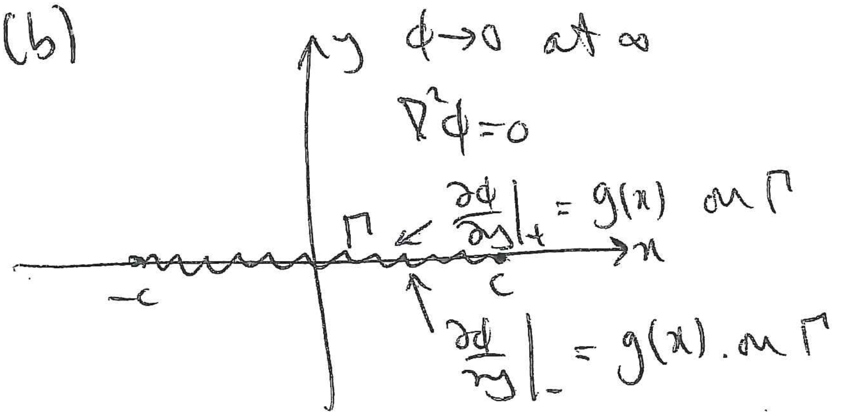
2nd integral $= -\frac{1}{2} \times 2\pi i \times$ residue

because of clockwise orientation. 2

$$\therefore w_-(t) = \frac{1}{2\pi i} \fint \frac{f(\zeta)\,d\zeta}{\zeta - t} - \frac{1}{2} f(t)$$ [bookwork]

$\boxed{6}$

(b)



[familiar example]

Let $\frac{\partial\phi}{\partial y} = \operatorname{Im}[w(z)]$; $\nabla^2\phi = 0 \Leftrightarrow$ $\boxed{w(z) \text{ holomorphic on } \mathbb{C}\setminus\bar{\Gamma}}$.

Then conditions become
$$\boxed{\begin{array}{l} w \to 0 \quad \text{as } z \to \infty \\ \operatorname{Im} w_+ = \operatorname{Im} w_- = g(x) \quad \text{on } \Gamma \end{array}}$$ 2

Write $\boxed{w(z) = \frac{1}{2\pi i}\int_{-c}^{c} \frac{f(\xi)\,d\xi}{\xi - z}}$ and then Plemelj gives

$$w_\pm(t) = \frac{1}{2\pi i} \int_{-c}^{c} \frac{f(\xi)\,d\xi}{\xi - t} \pm \frac{1}{2} f(t) \qquad \text{for } t \in (-c, c).$$

$$\therefore \; w_+(t) + w_-(t) = \frac{1}{\pi i} \int_{-c}^{c} \frac{f(\xi)\,d\xi}{\xi - t}$$

$$w_+(t) - w_-(t) = f(t) \in \mathbb{R}$$ 2

$\therefore$ $w_+(t) + w_-(t)$ is pure imaginary & $= 2i g(t)$

$$\therefore \; \boxed{-g(t) = \frac{1}{2\pi} \int_{-c}^{c} \frac{f(\xi)\,d\xi}{\xi - t}}$$ 2

6

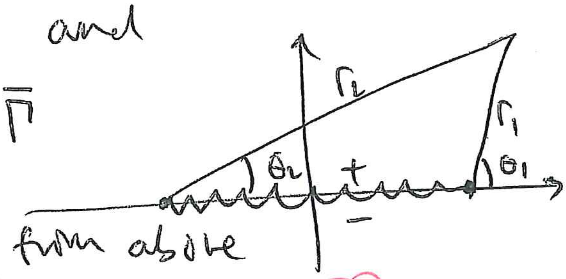
(c)

Define $\sqrt{z^2-c^2} = \sqrt{r_1 r_2}\, e^{i(\theta_1+\theta_2)/2}$ where $r_1 = |z-c|$, $r_2 = |z+c|$

$\theta_1 = \arg(z-c)$, $\theta_2 = \arg(z+c)$

and choose branch $-\pi < \theta_1, \theta_2 \leq \pi$

So branch cut is on $[-c, c]$ and $\sqrt{z^2-c^2}$ is holomorphic on $\mathbb{C}\setminus\overline{\Gamma}$



$\sqrt{z^2-c^2} \to i\sqrt{c^2-t^2}$ as $z \to t \in \Gamma$ from above

$\sqrt{z^2-c^2} \to -i\sqrt{c^2-t^2}$ as $z \to t \in \Gamma$ from below. 2

So let $\boxed{w(z)\sqrt{z^2-c^2} = W(z)}$

Then
$$W_+(t) - W_-(t) = i\sqrt{c^2-t^2}\left(w_+(t) + w_-(t)\right) = -2g(t)\sqrt{c^2-t^2}$$
$$W_+(t) + W_-(t) = i\sqrt{c^2-t^2}\left(w_+(t) - w_-(t)\right) = if(t)\sqrt{c^2-t^2}$$

2

$\therefore$ by Plemelj, $$W(z) = \frac{1}{2\pi i}\int_{-c}^{c} \frac{-2g(\xi)\sqrt{c^2-\xi^2}\,d\xi}{\xi - z} + \tilde{M}(z)$$

where $\tilde{M}(z)$ is holomorphic on $\mathbb{C}\setminus\{-c, c\}$ and pure imaginary on Γ

$$\therefore \boxed{w(z) = \frac{i}{\pi\sqrt{z^2-c^2}}\left[\int_{-c}^{c} \frac{g(\xi)\sqrt{c^2-\xi^2}\,d\xi}{\xi - z} + H(z)\right]}$$ 2

where H is real on Γ. $(H = -i\pi\tilde{M})$

[similar to lecture notes]

6

a).

$H(z)$ can only have isolated singularities at $z = \pm c$.

Since $\displaystyle\int_{-c}^{c} \frac{g(\xi)\sqrt{c^2-\xi^2}\,d\xi}{\xi \mp c} = -\int_{-c}^{c} g(\xi)\sqrt{\frac{c\pm\xi}{c\mp\xi}}\,d\xi$ is bounded

and given $\sqrt{z^2-c^2}\,W(z)$ is bounded as $z \to \pm c$, it follows that $H(z)$ is bounded as $z \to \pm c$, so any singularities there are removable and hence H is entire. 2

As $z \to \infty$, $\displaystyle W(z) \sim O\left(\frac{1}{z^2}\right) + \frac{iH(z)}{\pi z} = o(1/z)$ (given)

$\therefore H(z) \to 0$ as $z \to \infty$

$\therefore$ by Liouville, $H(z) \equiv 0$ 1 $\boxed{3}$

Now $W(z)$ can be bounded as $z \to \pm c$ only if

$$-H(c) = \int_{-c}^{c} \frac{g(\xi)\sqrt{c^2-\xi^2}\,d\xi}{\xi - c}$$

& $$-H(-c) = \int_{-c}^{c} \frac{g(\xi)\sqrt{c^2-\xi^2}\,d\xi}{\xi + c} \quad 2$$

$W(z) = O(1/z)$ as $z \to \infty$ $\Rightarrow$ $H \le O(1)$ as $z \to \infty$.

$\therefore$ (Liouville again) $H \equiv$ const.

$$\therefore H(c) - H(-c) = 0 = \int_{-c}^{c} g(\xi)\sqrt{c^2-\xi^2}\left[\frac{1}{\xi+c} - \frac{1}{\xi-c}\right]d\xi$$

$$0 = \int_{-c}^{c} \frac{g(\xi)\sqrt{c^2-\xi^2}}{\xi^2-c^2}(-2c)\,d\xi \Rightarrow \boxed{\int_{-c}^{c} \frac{g(\xi)\,d\xi}{\sqrt{c^2-\xi^2}} = 0}$$

2 $\boxed{4}$

Question 3

If $K_+(x) = \begin{cases} 0 & x<0 \\ e^{-3x} & x \geq 0 \end{cases}$ then

$$\bar{K}_+(k) = \int_0^\infty e^{(ik-3)x}\,dx = \frac{1}{3-ik} \quad \text{for } \operatorname{Im} k > -3$$

$$K_-(x) = \begin{cases} e^{x} & x<0 \\ 0 & x \geq 0 \end{cases}$$

$$\Rightarrow \bar{K}_-(k) = \int_{-\infty}^0 e^{(ik+1)x}\,dx = \frac{1}{1+ik} \quad \text{for } \operatorname{Im} k < 1$$

so $\boxed{\bar{K}(k) = \bar{K}_+(k) + \bar{K}_-(k) = \frac{1}{3-ik} + \frac{1}{1+ik}} = \frac{i}{k+3i} - \frac{i}{k-i}$

defined for $-3 < \operatorname{Im} k < 1$. May be analytically continued to $k \in \mathbb{C} \setminus \{-3i, i\}$. 4

$$\bar{g}_+(k) = \int_0^\infty x e^{ikx}\,dx \quad \text{converges for } \operatorname{Im} k > 0$$

$$= \left[-\frac{i}{k} x e^{ikx}\right]_0^\infty + \frac{i}{k}\int_0^\infty e^{ikx}\,dx$$

$$\therefore \boxed{\bar{g}_+(k) = -\frac{1}{k^2}}$$

for $\operatorname{Im} k > 0$. May be analytically continued to $k \in \mathbb{C} \setminus \{0\}$ 2

[New examples of familiar ideas] 6

(b) Integral equation (*):

$$\int_0^\infty K(x-t) f(t)\, dt = f(x) + x \qquad x \geq 0$$

Define f_+, g_+ and h_- as given, then

$$LHS = \int_{-\infty}^{\infty} K(x-t)\, f_+(t) = \begin{cases} f(x) + x & x \geq 0 \\ \int_0^\infty K(x-t) f(t)\, dt & x < 0 \end{cases}$$

i.e. $$\boxed{\int_{-\infty}^{\infty} K(x-t)\, f_+(t) = f_+(x) + g_+(x) + h_-(x)}$$ 2

Take F.T.; LHS is a convolution:

$$\bar{K}(k)\, \bar{f}_+(k) = \bar{f}_+(k) + \bar{g}_+(k) + \bar{h}_-(k)$$

i.e. $$\boxed{\left(\bar{K}(k) - 1\right) \bar{f}_+(k) - \bar{g}_+(k) = \bar{h}_-(k)}$$ 2

Given f continuous and $f(x) = O(x)$ as $x \to \infty$, it follows that $\bar{f}_+(k) = \int_0^\infty f(x) e^{ikx}\, dx$ is holomorphic in $\text{Im}(k) > 0$.

$$h_-(x) = e^{x} \int_0^\infty e^{-t} f(t)\, dt \qquad \text{for } x < 0$$

$$= O(e^{x}) \qquad \text{as } x \to -\infty$$ 2

$\therefore\ \bar{h}_-(k)$ is holomorphic in $\text{Im}(k) < 1$

6

[New applic^n of familiar ideas]

c) Now plug in expressions for $\bar{K}$ & $\bar{g}_+$:

$$\bar{K}(k) - 1 = \frac{i}{k+3i} - \frac{i}{k-i} - 1 = -\frac{k^2+2ik-1}{(k+3i)(k-i)}$$

$[i(k-i) - i(k+3i) - (k^2+2ik+3)]$ ✓

$$= -\frac{(k+i)^2}{(k+3i)(k-i)}$$

$$\therefore \; -\frac{(k+i)^2 \bar{f}_+}{(k+3i)(k-i)} + \frac{1}{k^2} = \bar{h}_-$$

$$\therefore \; \boxed{-\frac{(k+i)^2 \bar{f}_+}{(k+3i)} + \frac{k-i}{k^2} = (k-i)\bar{h}_-}$$ on $0 < \mathrm{Im}\, k <$

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LHS is holomorphic in $\mathrm{Im}\, k > 0$; RHS is holomorphic in $\mathrm{Im}\, k < 1$.

They are equal on a dense set of points so, by analytic continuation define an entire function $E(k)$, say.

Given $\bar{f}_+(k) = O(1/k)$ as $k \to \infty$, we see that

$E(k) \to$ constant as $k \to \infty$, $\therefore$ $E(k) \equiv C$ constant

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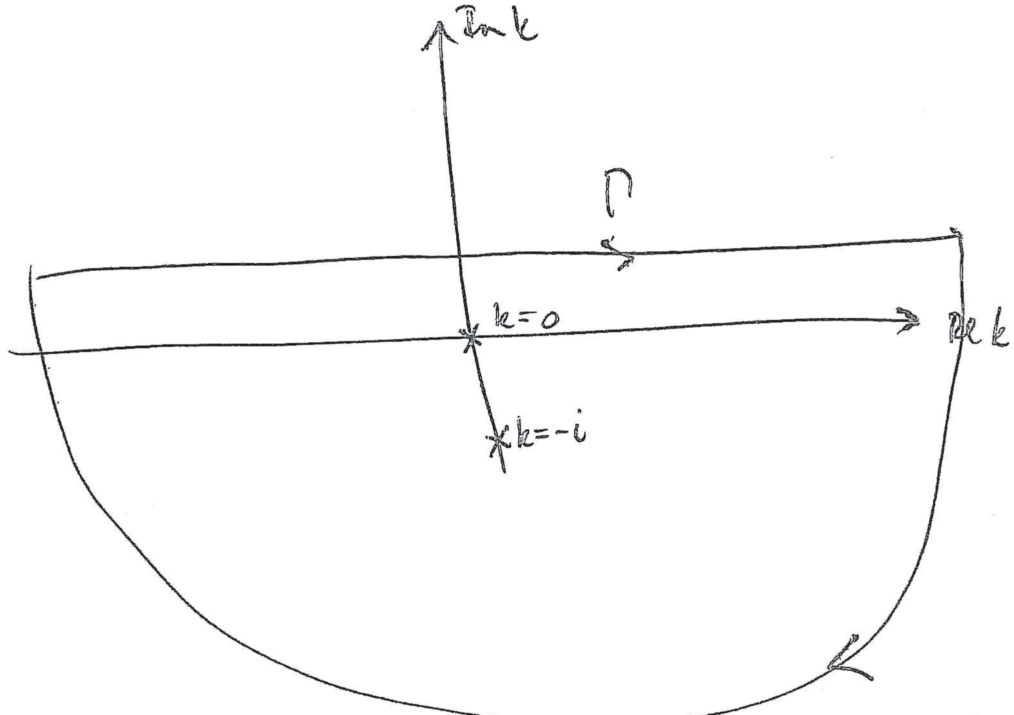
[New application of familiar ideas]

(d) $$\therefore \; \boxed{\bar{f}_+(k) = \frac{(k+3i)}{(k+i)^2}\left[\frac{k-i}{k^2} - C\right]}$$

$$\therefore \quad f_+(x) = \frac{1}{2\pi}\int_\Gamma \frac{(k+3i)}{(k+i)^2}\left[\frac{k-i}{k^2} - c\right]e^{-ikx}\,dk \quad ①$$

where Γ lies above singularities in $\bar{f}_+$:



Close contour as shown so $e^{-ikx} \to 0$ when $x > 0$ on curved bit of contour. ②

$$\therefore \quad f_+(x) = -i\,\text{Res}\left[\bar{f}_+(k)e^{-ikx}; 0\right] - i\,\text{Res}\left[\bar{f}_+(k)e^{-ikx}; -i\right]$$

[– signs because contour is clockwise]

Res at k=0 $= \dfrac{d}{dk}\left[\dfrac{(k+3i)(k-i)e^{-ikx}}{(k+i)^2}\right]\Bigg|_{k=0}$

$$= \left[\frac{(2k+2i)}{(k+i)^2} - \frac{2(k+3i)(k-i)}{(k+i)^3} - \frac{ix(k+3i)(k-i)}{(k+i)^2}\right]e^{-ikx}\Bigg|_{k=0}$$

$$= -2i - \frac{6}{-i} + ix.3 = (3x-8)i \quad ②$$

Res at $k=-i$ $= \frac{d}{dk}\left[(k+3i)\left(\frac{1}{k}-\frac{i}{k^2}-c\right)e^{-ikx}\right]\Big|_{k=-i}$

$$= \Big[\frac{1}{k}-\frac{i}{k^2}-c+(k+3i)\left(-\frac{1}{k^2}+\frac{2i}{k^3}\right)$$
$$-ix(k+3i)\left(\frac{1}{k}-\frac{i}{k^2}-c\right)\Big]e^{-ikx}\Big|_{k=}$$

$$=\left[i+i-c+2i(1+2)-ix(2i)(i+i-c)\right]e^{-x}$$

$$=\left[8i-c+2x(2i-c)\right]e^{-x}.$$

3

Altogether now:

$$\boxed{f_+(x) = 3x-8+\left(4x+8+Ci\,(2x+1)\right)e^{-x}}$$

which is equivalent to given result with $C \to \frac{\tilde{C}}{i}$.

[inversion integral standard but tricky!]

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