

SECOND PUBLIC EXAMINATION

Honour School of Mathematics Part C: Paper C5.6
Honour School of Mathematical and Theoretical Physics Part C: Paper C5.6
Master of Science in Mathematical and Theoretical Physics Part C: Paper C5.6

APPLIED COMPLEX VARIABLES

TRINITY TERM 2016

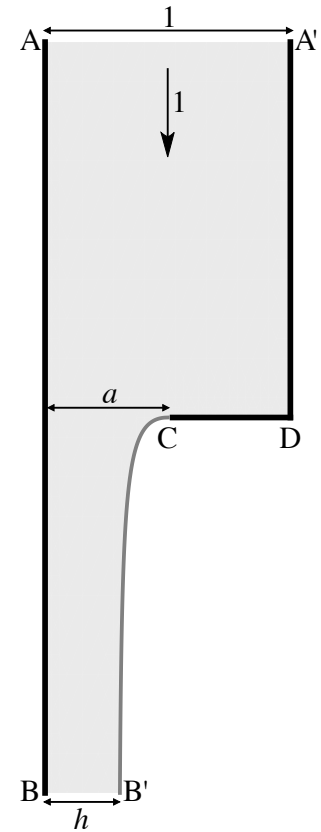
FRIDAY, 3 JUNE 2016, 2.30pm to 4.00pm

You may submit answers to as many questions as you wish but only the best two will count for the total mark.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

Do not turn this page until you are told that you may do so

1. Consider the illustrated two-dimensional steady flow. Fluid is fed at unit speed into a channel of unit thickness between vertical walls at AB and $A'D$. A wall CD perpendicular to $A'D$ forms a slot of thickness $a < 1$ through which the fluid flows as a jet, between the wall AB and a free surface $B'C$. The free surface detaches tangentially from the wall at C . The thickness $h < a$ of the jet at BB' far downstream of the slot is to be determined. Without loss of generality, define the streamfunction ψ to be zero on the wall AB and the velocity potential ϕ to be zero at C .



(a) [8 marks] Explain briefly why the flow speed must be equal to $1/h$ on the free surface $B'C$. Sketch the images of the shaded fluid domain in the potential w -plane and the hodograph w' -plane, indicating clearly the image of each of the labelled points.

(b) [9 marks] Show that the potential and hodograph planes are mapped to the same upper-half ζ -plane by the mappings

$$\zeta = e^{\pi w} + 1 = \left(\frac{1 - h^2}{1 + h^2} \right)^2 \left(\frac{1 - h^2 w'^2}{1 + h^2 w'^2} \right)^2.$$

Deduce that the average speed of the flow along the wall CD is given by

$$\int_C^D -u \, dx = \frac{2}{\pi} \log \left(\frac{1 + h^2}{2h} \right).$$

(c) [8 marks] If the velocity on the free surface is given by $w' = e^{-i\theta}/h$, show that $B'C$ is described parametrically by $z(\theta)$, where

$$\frac{dz}{d\theta} = \frac{2hH e^{i\theta} \tan \theta \sec^2 \theta}{\pi (1 + H \tan^2 \theta)}, \quad H = \left(\frac{1 - h^2}{1 + h^2} \right)^2.$$

Hence show that h is related to a by

$$a = h + \frac{1 - h^2}{\pi} \tan^{-1} \left(\frac{2h}{1 - h^2} \right).$$

2. (a) [11 marks] State the *Plemelj formulae*, making sure to define any notation that you use. Clearly define a branch of the multifunction

$$w(z) = \left(\frac{z-1}{z}\right)^{1/2} \log\left(\frac{z}{z-1}\right)$$

that is holomorphic on $\mathbb{C} \setminus \{x+iy : y=0, 0 \leq x \leq 1\}$, and such that $w(z)$ is real and positive when $z \in \{x+iy : y=0, x > 1\}$. By applying the Plemelj formulae to $w(z)$, show that

$$\int_0^1 \sqrt{\frac{1-\xi}{\xi}} \log\left(\frac{\xi}{1-\xi}\right) \frac{d\xi}{\xi-x} = \pi^2 \sqrt{\frac{1-x}{x}} \quad \text{for } 0 < x < 1.$$

- (b) Consider two-dimensional porous medium flow driven by a source of constant strength Q at the origin. The flow is described by a complex potential $w(z, t)$ which is holomorphic inside the time-dependent fluid domain $D(t) \setminus \{0\}$, with a specified logarithmic singularity at the origin, namely

$$w(z, t) = \frac{Q}{2\pi} \log z + O(1) \quad \text{as } z \rightarrow 0.$$

The free boundary conditions are

$$\operatorname{Re}(w) = 0, \quad \operatorname{Re}\left(\frac{\partial w}{\partial t}\right) + \left|\frac{dw}{dz}\right|^2 = 0 \quad \text{on } \partial D(t).$$

- (i) [7 marks] Suppose that $D(t)$ is the image in the z -plane of the unit disc $|\zeta| < 1$ under a time-dependent conformal map $z = F(\zeta, t)$ which satisfies the conditions

$$F(0, t) \equiv 0 \quad \text{and} \quad \frac{\partial F}{\partial \zeta}(0, t) > 0.$$

Show that $F(\zeta, t)$ must satisfy the differential equation

$$\operatorname{Re}\left[\zeta \frac{\partial F}{\partial \zeta} \overline{\frac{\partial F}{\partial t}}\right] = \frac{Q}{2\pi} \quad \text{on } |\zeta| = 1.$$

- (ii) [7 marks] Suppose that instead of a source, the flow is driven by a *dipole* at the origin, such that

$$w(z, t) = \frac{m}{z} + O(1) \quad \text{as } z \rightarrow 0,$$

where m is a given positive constant. Show that in this case the time-dependent mapping function from the unit disc $|\zeta| < 1$ to $D(t)$ satisfies the equation

$$\operatorname{Re}\left[\frac{\zeta^2}{1+\zeta^2} \frac{\partial F}{\partial \zeta} \overline{\frac{\partial F}{\partial t}}\right] + \mu(t) = 0 \quad \text{on } |\zeta| = 1,$$

where $\mu(t)$ is a positive real function, to be determined in terms of m and F .

3. (a) [5 marks] (i) Clearly define branches of the multifunctions $(k - i)^{1/2}$ and $(k + i)^{1/2}$ that are holomorphic and have positive real part for $-1 < \text{Im } k < 1$.
(ii) Let

$$f_+(x) = \begin{cases} \sin x & x > 0, \\ 0 & x < 0. \end{cases}$$

For what values of $k \in \mathbb{C}$ is the Fourier transform $\bar{f}_+(k)$ defined? Compute $\bar{f}_+(k)$. To what region of the complex k -plane may it be analytically continued?

- (b) [5 marks] The function $G(k)$ is holomorphic in a strip $\Omega = \{k \in \mathbb{C} : \alpha < \text{Im } k < \beta\}$ and satisfies $G(k) \rightarrow 0$ as $k \rightarrow \infty$ in Ω . Show that $G(k)$ may be decomposed as

$$G(k) = G_+(k) - G_-(k),$$

where $G_+(k)$ is holomorphic in $\text{Im } k > \alpha_1$ and $G_-(k)$ is holomorphic in $\text{Im } k < \beta_1$, for $\alpha < \alpha_1 < \beta_1 < \beta$. Give explicit formulae for $G_+(k)$ and $G_-(k)$, in terms of integrals along specified contours in the complex k -plane.

- (c) [4 marks] Using the definition of $(k - i)^{1/2}$ from part (a), let

$$G(k) = \frac{(k - i)^{1/2}}{k - c},$$

where c is an arbitrary real constant. Follow the procedure from part (b) and explicitly compute the integrals to evaluate $G_+(k)$ and $G_-(k)$ in this case, such that $G_+(k)$ is holomorphic in $\text{Im } k > 0$ and $G_-(k)$ is holomorphic in $\text{Im } k < 1$.

- (d) [11 marks] Suppose that $u(x, y)$ satisfies the partial differential equation

$$\nabla^2 u = u \quad \text{for } y > 0$$

subject to

$$\frac{\partial u}{\partial y}(x, 0) = 0 \quad \text{for } x < 0, \quad u(x, 0) = \sin x \quad \text{for } x > 0,$$

with $u(x, y) \rightarrow 0$ as $y \rightarrow \infty$, and $u(x, y)$ is bounded as $(x, y) \rightarrow (0, 0)$. Let

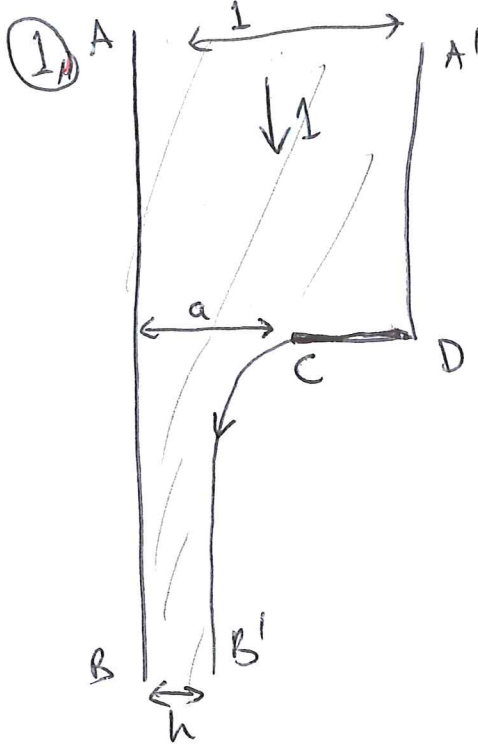
$$\frac{\partial u}{\partial y}(x, 0) = g_+(x) \quad \text{for } x > 0, \quad u(x, 0) = f_-(x) \quad \text{for } x < 0.$$

Assuming that $\bar{g}_+(k)$ is holomorphic in $\text{Im } k > 0$ and $\bar{f}_-(k)$ is holomorphic in $\text{Im } k < 1$, show that the corresponding Fourier transforms satisfy the equation

$$\frac{\bar{g}_+(k)}{(k + i)^{1/2}} + (k - i)^{1/2} \bar{f}_-(k) = \frac{(k - i)^{1/2}}{k^2 - 1}.$$

Assuming that $k^{1/2} \bar{g}_+(k)$ and $k \bar{f}_-(k)$ both tend to zero as $k \rightarrow \infty$, deduce that

$$\bar{g}_+(k) = \frac{(k + i)^{1/2}}{2^{3/4}} \left(\frac{e^{-i\pi/8}}{k - 1} - \frac{e^{-3i\pi/8}}{k + 1} \right).$$



(a) by mass conservation, flux out at $BB' = hV = 1 = \text{flux in at } AA'$
 $\therefore V \rightarrow 1/h \text{ at } BB'$

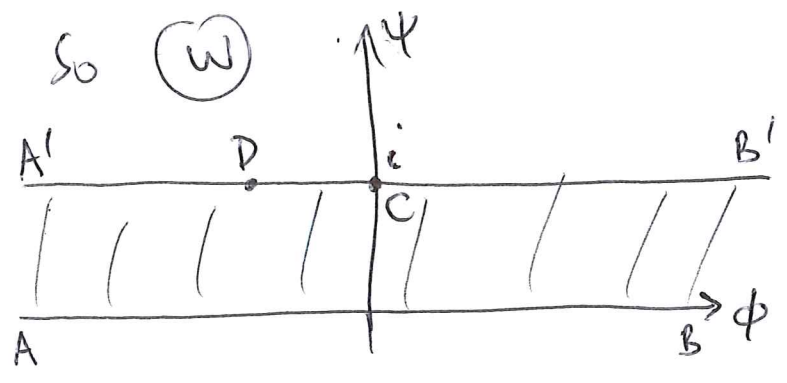
by Bernoulli, $p + \frac{1}{2}\rho|u|^2 = \text{const.}$
 $\therefore P = \text{const on free surface } B'C$
 $\therefore |u| = \text{const.} = 1/h \text{ on } B'C$

2) Standard result

Potential plane $\psi = 0$ on AB
 $\psi = 1$ on $A'DCB'$

At AA' , $(u, v) \rightarrow (0, -1) \Rightarrow \phi \sim -y \rightarrow -\infty$

At BB' , $(u, v) \rightarrow (0, -1/h) \Rightarrow \phi \sim -y/h \rightarrow +\infty$

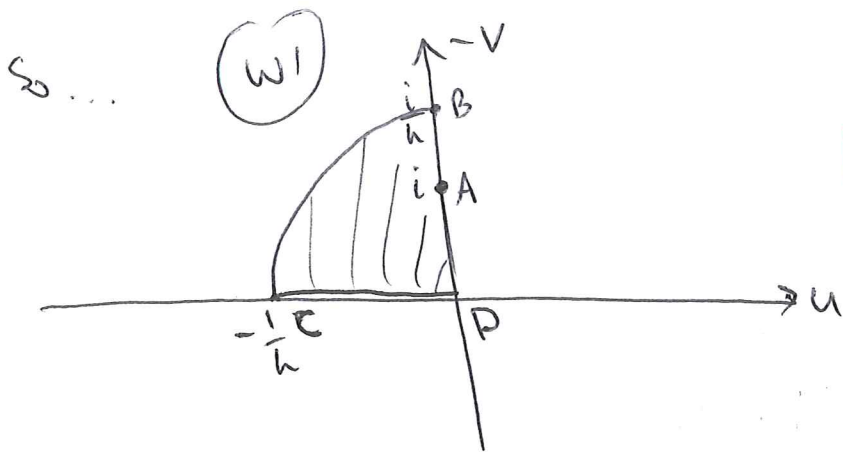


Given $\phi=0$ at C
 $\Rightarrow w = i$ at C

D is at $i - \delta$ where δ is to be determined.

3) Standard approach

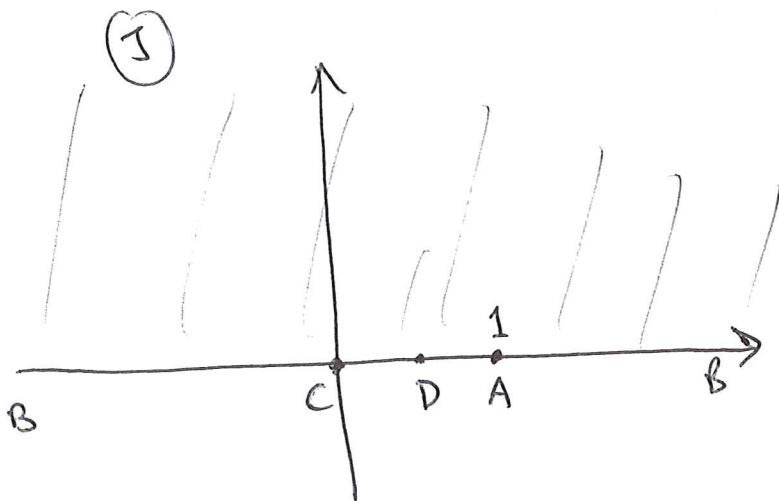
- W-plane $w' = u - iv \rightarrow i$ at AA'
 $\rightarrow i/h$ at BB'
 $\rightarrow 0$ at D (angle $< \pi$)
- On AB , $u=0$, v increases from 1 to $1/h$.
 - On $A'D$, $u=0$, v decreases from 1 to 0 .
 - On DC , $v=0$, u increases from 0 to $1/h$.
 - On CB' , $|w'| = 1/h$, v decreases from 0 to $-1/h$
 u decreases from $-1/h$ to 0



3 Standard approach

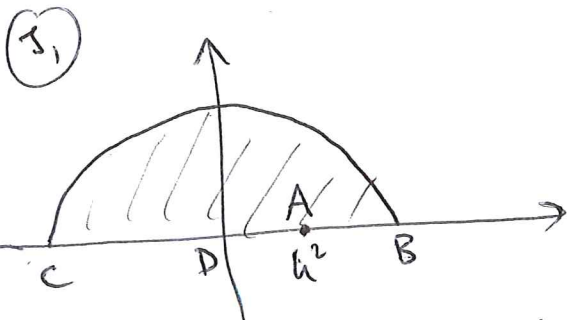
(b) let $S = e^{\pi w} + 1$ then shaded region in w-plane is mapped to $\text{Im}(S) > 0$, with

- $A \rightarrow 1$
- $B \rightarrow \infty$
- $C \rightarrow 0$
- $D \rightarrow 1 - e^{-\pi \delta} \in (0, 1)$



2 Standard approach

Let $S_1 = -h^2(w')^2$ so $A \mapsto S_1 = h^2 \in (0, 1)$



- $B \mapsto S_1 = 1$
- $C \mapsto S_1 = -1$
- $D \mapsto S_1 = 0$

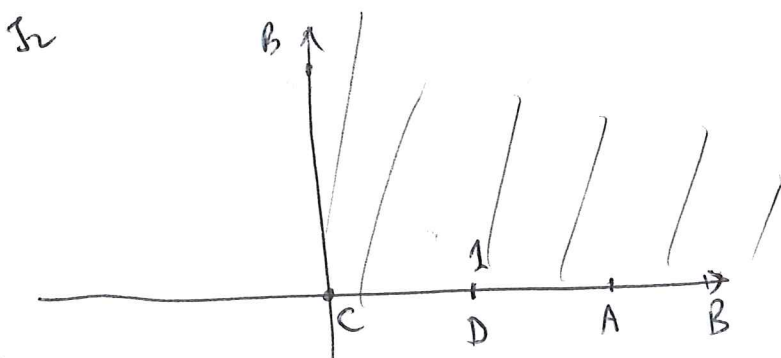
Now Möbius: $S_2 = \frac{1 + S_1}{1 - S_1}$

$A \mapsto \frac{1 + h^2}{1 - h^2} > 1$

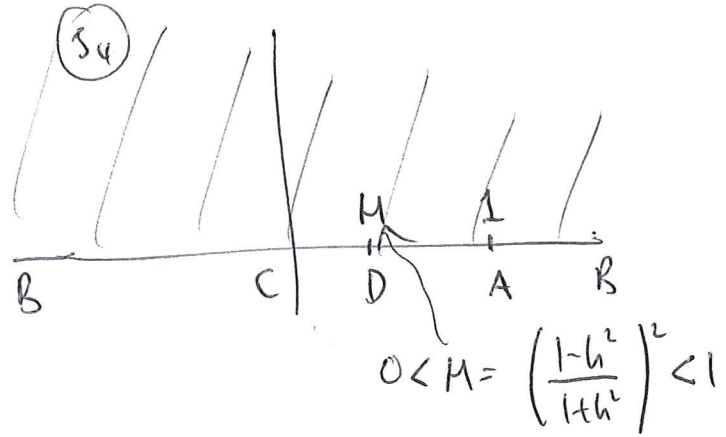
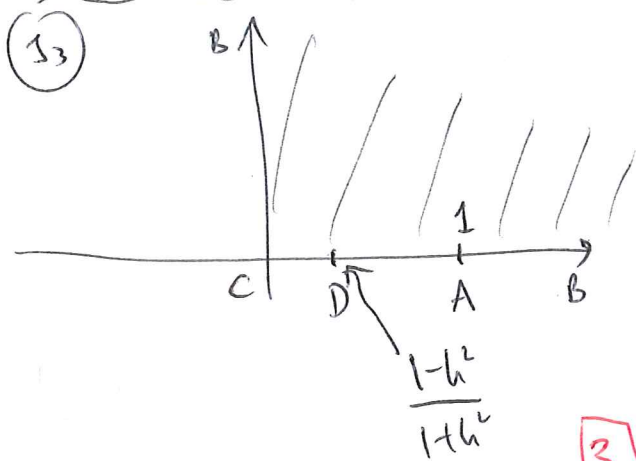
$B \mapsto \infty$

$C \mapsto 0$

$D \mapsto 1$



$$z_3 = z_2 \left(\frac{1-h^2}{1+h^2} \right) \text{ (just a scaling) ; then square } z_4 = z_3^2$$



3 standard example

Note that the shaded fluid domain is the same upper half-plane in the z and z_4 -plane. Three points on the boundary A, B, C are mapped to the same point on $\text{Im } z = 0$.

By uniqueness of conformal mapping (Riemann) we must

have

$$z = z_4 \Rightarrow e^{\pi w} + 1 = M \left(\frac{1 - h^2(w')^2}{1 + h^2(w')^2} \right)^2$$

2 standard approach

We also get that the position of D must be the same:

$$M = 1 - e^{-\pi \delta} \Rightarrow e^{-\pi \delta} = 1 - \frac{(1-h^2)^2}{(1+h^2)^2} = \frac{4h^2}{(1+h^2)^2}$$

$$\therefore -\pi \delta = 2 \log \left(\frac{2h}{1+h^2} \right)$$

$$\therefore \delta = \frac{2}{\pi} \log \left(\frac{1+h^2}{2h} \right)$$

$$\text{So } \int_c^D -u \, dx = [-\phi]_c^D = \delta = \frac{2}{\pi} \log \left(\frac{1+h^2}{2h} \right)$$

2 New example

(c) Let $w' = e^{-i\theta}/h$

$$e^{\pi w} + 1 = M \left(\frac{1 - e^{-2i\theta}}{1 + e^{2i\theta}} \right)^2 = -M \tan^2 \theta$$

Differentiate w.r.t. θ :

$$\pi e^{\pi w} w' \frac{dz}{d\theta} = -2M \tan \theta \sec^2 \theta$$

$$\therefore \pi \left[-1 - M \tan^2 \theta \right] \frac{e^{-i\theta}}{h} \frac{dz}{d\theta} = -2M \tan \theta \sec^2 \theta$$

$$\Rightarrow \boxed{\frac{dz}{d\theta} = \frac{2hM \tan \theta \sec^2 \theta e^{i\theta}}{\pi (1 + M \tan^2 \theta)}}$$

[3] standard approach

Take real part:

$$\frac{dh}{d\theta} = \frac{2hM}{\pi} \frac{\tan \theta \sec \theta}{1 + M \tan^2 \theta} = \frac{2hM}{\pi} \frac{\sin \theta}{\cos^2 \theta + M(1 - \cos^2 \theta)}$$

At C, $x = a$, $\theta = \pi$

At B', $x = h$, $\theta = \frac{3\pi}{2}$

$$\therefore \int_{\pi}^{\frac{3\pi}{2}} \frac{dh}{d\theta} d\theta = h - a = \frac{2hM}{\pi} \int_{\pi}^{\frac{3\pi}{2}} \frac{\sin \theta d\theta}{H + (1-M)\cos^2 \theta}$$

Let $\cos \theta = -c \quad \therefore \sin \theta d\theta = dc \quad ; \quad c = \sqrt{\frac{H}{1-M}} \tan t$

$$h - a = -\frac{2hM}{\pi(1-M)} \int_0^1 \frac{dc}{c^2 + \frac{M}{1-M}} = \frac{-2hM}{\pi(1-M)} \sqrt{\frac{1-M}{H}} \tan^{-1} \sqrt{\frac{1-M}{H}}$$

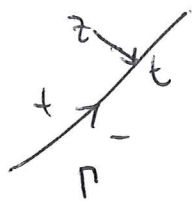
Recall $1-M = \frac{4h^2}{(1+h^2)^2} \quad ; \quad \frac{1-M}{H} = \frac{4h^2}{(1+h^2)^2}$

$$\therefore a - h = \frac{2h}{\pi} \frac{(1-h^2)}{2h} \cdot \tan^{-1} \left(\frac{2h}{1-h^2} \right)$$

$$\therefore a = h + \frac{(1-h^2)}{\pi} \tan^{-1} \left(\frac{2h}{1-h^2} \right)$$

5 New example

(1a) Plemelj formulae: for $w(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta) d\zeta}{\zeta - z}$



limit from left side (relative to direction of Γ)

as $z \rightarrow t \in \Gamma$ is $w_+(t)$

limit from right = $w_-(t)$

then

$$w_{\pm}(t) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta) d\zeta}{\zeta - t} \pm \frac{1}{2} f(t)$$

where, with $\gamma_{\epsilon} = \Gamma \setminus D(t; \epsilon)$

$$\int_{\Gamma} \frac{f(\zeta) d\zeta}{\zeta - t} = \lim_{\epsilon \rightarrow 0} \int_{\Gamma \setminus \gamma_{\epsilon}} \frac{f(\zeta) d\zeta}{\zeta - t} \quad \text{[3] Bolderwork}$$

let $z-1 = r_1 e^{i\theta_1}$, $z = r_2 e^{i\theta_2}$; then

$$\begin{aligned} w(z) &= \left(\frac{z-1}{z}\right)^{1/2} \log\left(\frac{z}{z-1}\right) \\ &= \sqrt{\frac{r_1}{r_2}} e^{i(\theta_1 - \theta_2)/2} \left[\log\left(\frac{r_2}{r_1}\right) + i(\theta_2 - \theta_1) \right] \end{aligned}$$

[2] standard approach.

with $-\pi \leq \theta_1, \theta_2 \leq \pi$

then $w(z)$ is holomorphic

on $\mathbb{C} \setminus [0, 1]$ with

(7)

$$\begin{aligned} \theta_1 = \pi, \theta_2 = 0 \Rightarrow w_+(x) &= i \sqrt{\frac{1-x}{x}} \left[\log\left(\frac{x}{1-x}\right) - i\pi \right] \\ \theta_1 = -\pi, \theta_2 = 0 \Rightarrow w_-(x) &= -i \sqrt{\frac{1-x}{x}} \left[\log\left(\frac{x}{1-x}\right) + i\pi \right] \end{aligned}$$

and when $z = x \in \mathbb{R} > 1$, $\theta_1 = \theta_2 = 0$ so

$$w(x) = \sqrt{\frac{x-1}{x}} \log\left(\frac{x}{x-1}\right) > 0 \quad \text{since } \frac{x}{x-1} > 1.$$

② standard approach.

Write $w(z) = \frac{1}{2\pi i} \int_0^1 \frac{f(\zeta) d\zeta}{\zeta - z}$

to get $w_{\pm}(x) = \frac{1}{2\pi i} \int_0^1 \frac{f(\zeta) d\zeta}{\zeta - x} \pm \frac{1}{2} f(x)$

$$= \pi \sqrt{\frac{1-x}{x}} \pm i \sqrt{\frac{1-x}{x}} \log\left(\frac{x}{1-x}\right)$$

$$\therefore f(x) = 2i \sqrt{\frac{1-x}{x}} \log\left(\frac{x}{1-x}\right)$$

and $\frac{1}{2\pi i} \int_0^1 2i \sqrt{\frac{1-\zeta}{\zeta}} \log\left(\frac{\zeta}{1-\zeta}\right) \frac{d\zeta}{\zeta - x} = \pi \sqrt{\frac{1-x}{x}}$

ie. $\int_0^1 \sqrt{\frac{1-\zeta}{\zeta}} \log\left(\frac{\zeta}{1-\zeta}\right) \frac{d\zeta}{\zeta - x} = \pi^2 \sqrt{\frac{1-x}{x}}$

for $x \in (0, 1)$.

④ New example.

(26) (i) let potential in the z -plane be

$$W(z, t) = w(F(z, t), t)$$

then ① $W(z, t)$ is holomorphic in $0 < |z| < 1$

As $z \rightarrow 0$, $F(z, t) = z \frac{\partial F}{\partial z}(0, t) + O(z^2)$ [Recall $\frac{\partial F}{\partial z}(0, t) \in \mathbb{R}^1$]

so $W(z, t) \sim \frac{Q}{2\pi} \log \left(z \frac{\partial F}{\partial z}(0, t) \right) + O(1)$

\Rightarrow ② $W(z, t) \sim \frac{Q}{2\pi} \log z + O(1)$ as $z \rightarrow 0$

Also ③ $\operatorname{Re}[W(z, t)] = 0$ on $|z| = 1$ [which is the pre-image of ∂D]

From ①-③, we get $W(z, t) = \frac{Q}{2\pi} \log z$

For kinetic BC, Note

$$\frac{\partial W}{\partial s} = \frac{\partial F}{\partial s} \frac{\partial w}{\partial t}, \quad \frac{\partial W}{\partial t} = \frac{\partial F}{\partial t} \frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}$$

so $\operatorname{Re} \left[\frac{\partial W}{\partial t} - \frac{\frac{\partial W}{\partial s} \frac{\partial F}{\partial t}}{\frac{\partial F}{\partial s}} \right] + \frac{|\frac{\partial W}{\partial s}|^2}{|\frac{\partial F}{\partial s}|^2} = 0$ on $|z| = 1$

But $\operatorname{Re}[W] = 0$ on $|z| = 1 \Rightarrow \operatorname{Re} \left(\frac{\partial W}{\partial t} \right) = 0$ on $|z| = 1$.

Rearrange to $\operatorname{Re} \left[\frac{\frac{\partial F}{\partial s} \overline{\frac{\partial F}{\partial t}}}{\frac{\partial W}{\partial s}} \right] = 1 \Rightarrow \operatorname{Re} \left[z \frac{\partial F}{\partial s} \overline{\frac{\partial F}{\partial t}} \right] = \frac{Q}{2\pi}$
on $|z| = 1$

□ standard example.

(ii) If $w(z,t) = \frac{m}{z} + O(1)$ as $z \rightarrow 0$

then we get $W(\zeta, t) \sim \frac{m}{\int \frac{\partial F(0,t)}{\partial \zeta}} + O(1)$

$W(\zeta, t) \sim \frac{\mu(t)}{\int} + O(1)$

where $\mu = \frac{m}{\int \frac{\partial F(0,t)}{\partial \zeta}}$ & recall that $\frac{\partial F(0,t)}{\partial \zeta} > 0$

We also have $\cdot W(\zeta, t)$ holomorphic on $0 < |\zeta| < 1$
 and $\cdot \operatorname{Re} W(\zeta, t) = 0$ on $|\zeta| = 1$.

The solution now is $W(\zeta, t) = \mu(t) \left(\frac{1}{\zeta} - \zeta \right)$

Since $\operatorname{Re}[W(\zeta, t)] = 0$ on $|\zeta| = 1$, we still have

$\operatorname{Re} \left[\frac{\partial W}{\partial t} \right] = 0$ on $|\zeta| = 1$ and as before

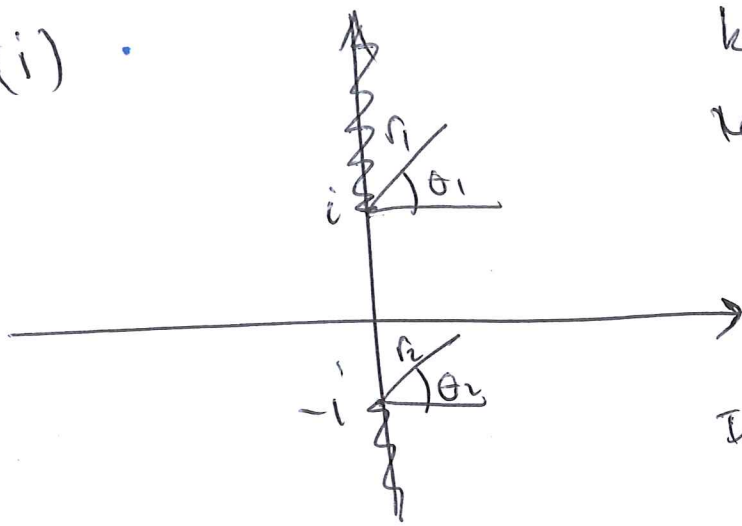
Kirchhoff BC reads $\operatorname{Re} \left[\frac{\frac{\partial F}{\partial \zeta} \overline{\frac{\partial F}{\partial t}}}{\frac{\partial W}{\partial \zeta}} \right] = 1$ on $|\zeta| = 1$

$\Rightarrow \operatorname{Re} \left[\frac{-\zeta^2 \frac{\partial F}{\partial \zeta} \overline{\frac{\partial F}{\partial t}}}{\mu(1+\zeta^4)} \right] = 1$ on $|\zeta| = 1$

ie. $\operatorname{Re} \left[\frac{\zeta^2}{1+\zeta^2} \frac{\partial F}{\partial \zeta} \overline{\frac{\partial F}{\partial t}} \right] + \mu = 0$ on $|\zeta| = 1$

\square New example

3a(i)



$$k-i = r_1 e^{i\theta_1}$$

$$\text{then } \boxed{(k-i)^{1/2} = \sqrt{r_1} e^{i\theta_1/2}}$$

$$\text{with } -\frac{3\pi}{2} < \theta_1 < \frac{\pi}{2}$$

In $\text{Im } k < 1$ we have

$$-\pi < \theta_1 < 0$$

$$\Rightarrow \text{Re } (k-i)^{1/2} = \sqrt{r_1} \cos\left(\frac{\theta_1}{2}\right) > 0$$

Similarly $\underline{(k+i)^{1/2} = \sqrt{r_2} e^{i\theta_2/2}}$ with $-\frac{\pi}{2} < \theta_2 < \frac{3\pi}{2}$

In $\text{Im } k > 1$ we have $0 < \theta_2 < \pi$

$$\text{so } \text{Re } (k+i)^{1/2} = \sqrt{r_2} \cos\left(\frac{\theta_2}{2}\right) > 0$$

[2] standard approach

(ii) $\widehat{f}_+(k) = \int_0^\infty \sin x e^{ikx} dx$ exists for $\text{Im } k > 0$

$$= \frac{1}{2i} \int_0^\infty e^{i(k+1)x} - e^{i(k-1)x} dx$$

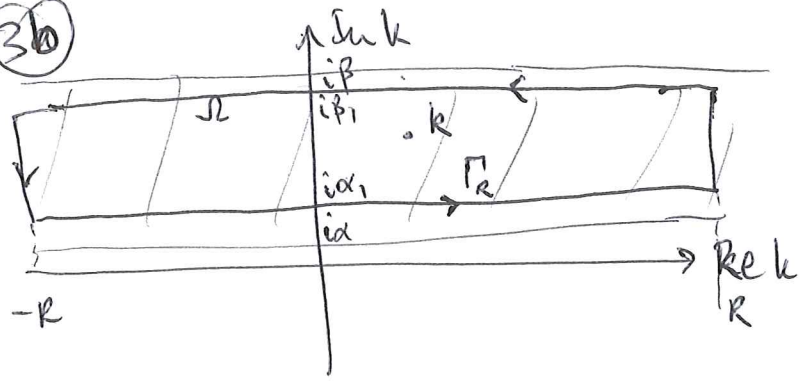
$$= \frac{1}{2i} \left[\frac{e^{i(k+1)x}}{i(k+1)} - \frac{e^{i(k-1)x}}{i(k-1)} \right]_0^\infty$$

$$= \frac{1}{2} \left[\frac{1}{k+1} - \frac{1}{k-1} \right] = \underline{\underline{\frac{-1}{k^2-1}}}$$

Can be analytically continued onto $\mathbb{C} \setminus \{\pm 1\}$

[3] standard approach.

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Let P_R be rectangular contour enclosing $k \in \Omega$ with sides $\text{Re}(k) = \pm R$
 $\text{Im}(k) = \alpha_1, \beta_1$
 with $\alpha < \alpha_1 < \text{Im} k < \beta_1 < \beta$.

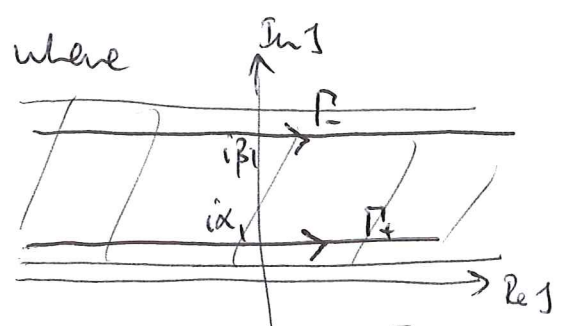
By Cauchy's Integral formula,

$$G(k) = \frac{1}{2\pi i} \oint_{P_R} \frac{G(z) dz}{z - k}$$

As $R \rightarrow \infty$, contributions from ends of rectangles $\rightarrow 0$ (since $G \rightarrow 0$), so we get

$$G(k) = G_+(k) - G_-(k)$$

$$G_{\pm}(k) = \frac{1}{2\pi i} \int_{P_{\pm}} \frac{G(z) dz}{z - k}$$



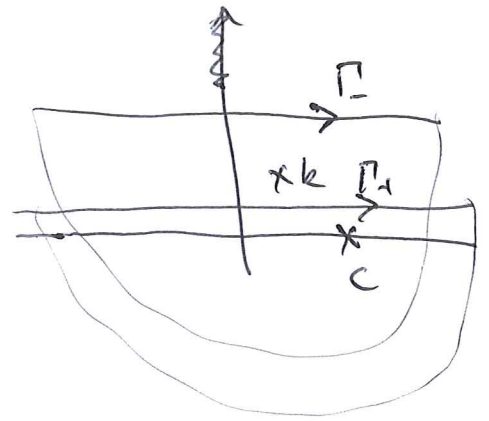
where
 $\text{Im} z = \alpha_1$ on P_+
 $\text{Im} z = \beta_1$ on P_-

So $G_{\pm}(k)$ is holomorphic for $k \notin P_{\pm}$ and in particular

$$\begin{aligned} G_+(k) & \text{ is holomorphic in } \text{Im} k > \alpha_1, \\ G_-(k) & \text{ " " " " } \text{Im} k < \beta_1 \end{aligned}$$

5) bookwork

(3c) Now compute $\int_{\Gamma_{\pm}} \frac{(s-i)^{1/2}}{(s-c)(s-k)} ds$



Close integration contours in lower half-plane:

$$G_+(k) = -\text{Res}(s=c)$$

$$G_-(k) = -\text{Res}(s=c) - \text{Res}(s=k)$$

$$\therefore G_+(k) = -\frac{(c-i)^{1/2}}{c-k}, \quad G_-(k) = -\frac{(c-i)^{1/2}}{c-k} - \frac{(k-i)^{1/2}}{k-c}$$

$$\therefore \boxed{G_+(k) = \frac{(c-i)^{1/2}}{k-c}, \quad G_-(k) = \frac{(c-i)^{1/2} - (k-i)^{1/2}}{k-c}}$$

[4] New example.

Now we have $\nabla^2 u = u \quad y > 0$

$\frac{\partial u}{\partial y} = g_+(k), \quad u = f_+(k) + f_-(k) \quad \text{on } y=0$

Fourier Transform :
$$\begin{cases} \frac{\partial^2 \bar{u}}{\partial y^2} = (k^2+1) \bar{u} & y > 0 \\ \bar{u} = \bar{f}_+ + \bar{f}_-, \quad \frac{\partial \bar{u}}{\partial y} = \bar{g}_+ & y = 0 \\ \bar{u} \rightarrow 0 & y \rightarrow \infty \end{cases}$$

$$\Rightarrow \bar{u} = (\bar{f}_+ + \bar{f}_-) e^{-(k^2+1)^{1/2} y}$$

$$\& \bar{g}_+ = -(\bar{f}_+ + \bar{f}_-) (k^2+1)^{1/2}$$

where $\text{Re} (k^2+1)^{1/2} > 0$

$(k^2+1)^{1/2} = (k+i)^{1/2} (k-i)^{1/2}$
defined as $i\pi$.

$$\therefore \boxed{\frac{\bar{g}_+(k)}{(k+i)^{1/2}} + (k-i)^{1/2} \bar{f}_-(k) = \frac{(k-i)^{1/2}}{k^2-1}}$$

[3] New example.

Need sum decomposition of RHS: (using part (b))

$$\begin{aligned} \text{RHS} &= \frac{(k-i)^{1/2}}{2} \left[\frac{1}{k-1} - \frac{1}{k+1} \right] \\ &= \frac{1}{2} \left\{ \frac{(1-i)^{1/2}}{k-1} + \frac{(k-i)^{1/2} - (1-i)^{1/2}}{k-1} - \frac{(-1-i)^{1/2}}{k+1} \right. \\ &\quad \left. + \frac{(-1-i)^{1/2} - (k-i)^{1/2}}{k+1} \right\} \end{aligned}$$

So rearrange to:

$$\begin{aligned} \frac{\bar{g}_+(k)}{(k+i)^{1/2}} + \frac{1}{2} \left\{ \frac{(-1-i)^{1/2}}{k+1} - \frac{(1-i)^{1/2}}{k-1} \right\} &= - (k-i)^{1/2} \bar{f}_-(k) \\ + \frac{1}{2} \left\{ \frac{(k-i)^{1/2} - (1-i)^{1/2}}{k-1} + \frac{(-1-i)^{1/2} - (k-i)^{1/2}}{k+1} \right\} &= E(k) \end{aligned}$$

3 New example

By given assumptions, LHS is holomorphic in $\text{Im } k > 0$

and LHS $\rightarrow 0$ as $k \rightarrow \infty$ with $\text{Im } k > 0$

RHS is holomorphic in $\text{Im } k < 1$ [the poles at $k = \pm 1$ have been removed]

and RHS $\rightarrow 0$ as $k \rightarrow \infty$ in $\text{Im } k < 1$

So together, they define an entire function $E(k)$.

Since $E(k) \rightarrow 0$ as $k \rightarrow \infty$, Liouville's Th^m

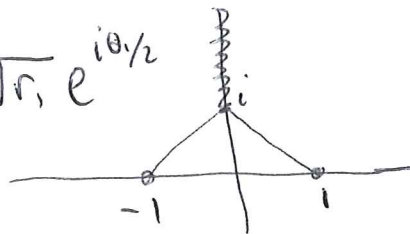
$$\Rightarrow \underline{E(k) \equiv 0} \Rightarrow \bar{g}_+(k) = \frac{(k+i)^{1/2}}{2} \left[\frac{(1-i)^{1/2}}{k-1} - \frac{(-1-i)^{1/2}}{k+1} \right]$$

3 standard argument

From definition of $(k-i)^{1/2} = \sqrt{r_1} e^{i\theta/2}$

at $k=1$, $r_1 = \sqrt{2}$, $\theta_1 = -\frac{\pi}{4}$

at $k=-1$, $r_1 = \sqrt{2}$, $\theta_1 = -\frac{3\pi}{4}$



$$\therefore (1-i)^{1/2} = 2^{1/4} e^{-i\pi/8}, \quad (-1-i)^{1/2} = 2^{1/4} e^{-3i\pi/8}$$

$$\& \boxed{\bar{g}_+(k) = \frac{(k+i)^{1/2}}{2^{3/4}} \left[\frac{e^{-i\pi/8}}{k-1} - \frac{e^{-3i\pi/8}}{k+1} \right]}$$

② standard approach.